# Initial Transient Response of Oscillators with Long Settling Time

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***Abstract*—The initial transient response of oscillators with high quality factor Q such as quartz crystal oscillators is orders of magnitudes larger than the period of oscillation. Therefore numerical solution by standard techniques of the underlying system of ordinary differential algebraic equations (DAEs) resulting from Kirchhoff's current and voltage laws is run time inefficient. In this paper numerical techniques for the calculation of the initial transient response and steady state solution are investigated. The efficiency results from reformulating the underlying system of ordinary DAEs by a suitable system of partial DAEs, known as multirate PDE, and from suitable finite difference time domain (FDTD) methods with small numerical dissipation of energy. Unlike Harmonic Balance the waveforms are free of spurious oscillations, caused by the non-compactness of the trigonometric polynomials.**

***Index Terms*—oscillator simulation, quartz crystal oscillators, initial transient response and steady state, multirate PDE method, trigonometric BDF methods, optimal estimation of instantaneous frequency, Hilbert transformation**

## I. Introduction

The simulation of oscillators with a high Q factor belongs to the severest problems in circuit simulation due to long settling time. On the one hand, the initial transient response is orders of magnitude larger than the period of oscillation. The sample frequency on the other hand must be chosen much smaller than the Nyquist rate or vice versa the time step $\Delta t \ll T$, where $T$ is the period of oscillation, $f_0 = \frac{1}{T}$ the free running, and $\omega_0 = 2\pi f_0$ the angular frequency. The run time is therefore prohibitively long, even for small circuits. Widely used integration methods such as Gear's multistep backward differentiation formulas (BDF) [1] lead to erroneous steady states caused by a numerical dissipation or loss of energy [2]. The trapezoidal method on the one hand does not exhibit numerical damping or dissipation, on the other hand it is not L-stable and may result in erroneous and spurious oscillations. Harmonic Balance (HB) does not exhibit numerical damping, because the numerical differentiation operator is purely imaginary. However in some cases, as shown in Section IV, spurious oscillations have been observed too.

Several attempts have been made for the simulation of ordinary differential equations with highly oscillatory solutions applying special integration formulas [3]–[6]. The class of envelope following techniques or multirate methods try to overcome the bottleneck due to Nyquist's theorem by separating the slowly varying waveforms, i.e. the envelopes, from the fast varying oscillation [4], [6]–[13].

The multirate method employed here is based on the reformulation of the underlying system of DAEs as a system of suitable partial DAEs [4], [6], [10] with an estimate of the instantaneous frequency as an additional variable (see Section II). The instantaneous frequency is well defined employing the Hilbert transformation. The computation of the Hilbert transform is unfortunately burdensome. Here, methods are presented which give near optimal estimates. Moreover, the steady state solution is not unique. Instead, any time shifted solution solves the underlying DAE too. There exist several techniques for the calibration of the solution. One method is to employ the Poincaré map technique (see e.g. [7]). Alternatively one can force the solution being orthogonal to a reference waveform w.r.t. a given inner product [4]. The method used here is well suited for envelope analysis as it calculates the instantaneous frequency by minimizing a suitable distance measure in the $L_2$ sense [14].

The effect of numerical dissipation or loss of energy is treated in [2]. This unwanted effect is a severe problem in simulating high Q oscillators. One technique to overcome the problem is to expand the oscillatory waveform in a Fourier series, which leads to Harmonic Balance (HB) type methods [6]. Alternatively one can employ high order single step methods [7], [15] with extremely low discretization error or trigonometric splines [16]. In this paper modified BDF methods are employed.

## II. The envelope technique

Envelope methods [4], [6]–[13] are a class of techniques to overcome Nyquist's barrier by introducing different time scales for the envelope(s) and the oscillatory signals. One method [17], [18] reformulates the underlying ordinary DAEs by a system of partial DAEs. Several modifications of this method have been proposed [11], [13], [19]. The technique employed here introduces the instantaneous frequency as an additional unknown [4], [10], [14].

As oscillators are autonomous systems, the modified nodal analysis (MNA) leads to an initial value problem

$$f(x(t),\dot{x}(t)) = i(x(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\,q(x(t)) = 0,\ x(0) = x_0,\ f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \tag{1}$$

Instead of solving (1), the DAE is reformulated by a system of partial differential equations (PDEs)

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$$f\left(\hat{x}(\tau,t_1),\frac{\partial}{\partial\tau}\hat{x}(\tau,t_1),\frac{\partial}{\partial t_1}\hat{x}(\tau,t_1)\right)=$$
$$i(\hat{x}(\tau,t_1))+\left(\frac{\partial}{\partial\tau}+\frac{\mathrm{d}(\tau\,\omega_0(\tau))}{\mathrm{d}\tau}\frac{\partial}{\partial t_1}\right)q(\hat{x}(\tau,t_1))=0$$
$$\hat{x}(0,t_1)=x_0(t_1),\quad x_0(0)=x_0 \tag{2}$$

Periodicity is assumed along the $t_1$ direction, i.e. $\hat{x}(\tau,t_1)=\hat{x}(\tau,t_1+2\pi)$, capturing the oscillatory behavior. A solution is therefore calculated along the domain or strip $\Omega$

$$\Omega:=\{(\tau,t_1)\,|\,\tau\in\mathbb{R}^+,\quad t_1\in[0,2\pi)\} \tag{3}$$

The fast varying oscillation is represented along the time scale $t_1$, whereas the smooth envelope is modeled along the time scale $\tau$. The solution of the underlying ordinary DAE system (1) is obtained along the specific characteristic curve ($\tau=t$, $t_1=\omega_0(t)\,t$), i.e. $x(t)=\hat{x}(t,\omega_0(t)\,t)$ [4], [14]. $\hat{x}$ has to be extended periodically outside the domain $\Omega$. Equation (2) is an under-determined system because $\omega_0(\tau)$, the instantaneous angular frequency, is not specified. An optimal choice, where a smooth envelope curve is obtained, has been reported in [14].

### A. Initialization of the multirate PDE

The initial conditions of the initial value problem (2) have to be specified, where $x_0(t_1)$ is $2\pi$-periodic by construction. It is assumed that the operating point is unstable so that a small perturbation leads to a growing amplitude. The perturbation has to be performed such that the envelope remains smooth making the multirate PDE technique efficient. A solution of this problem is provided by a theorem of Grobman and Hartman [20]. The theorem emphasizes the fact that orbits of a nonlinear ODE system are similar to the orbits of the linearized system near the unstable equilibrium point, known as small signal analysis. Therefore the initialization is performed in two steps, calculation of the unstable operating point first and the solution manifold of the linearized system second. Let $\bar{x}_0$ be an operating point of (1), i.e. $i(\bar{x}_0)=0$. Furthermore let $G:=i_x(x)\big|_{x=\bar{x}_0}$ and $C:=q_x(x)\big|_{x=\bar{x}_0}$ be the Jacobian matrices evaluated at $\bar{x}_0$. One obtains the homogeneous linear differential equation

$$G\,x+C\,\dot{x}=0 \tag{4}$$

In what follows it is assumed that the oscillator DAE is of index 0 or 1. Index investigations of oscillator DAEs back the usefulness of that assumption. A solution of the generalized eigenvalue problem

$$G\,Y+C\,Y\,\Lambda=0 \tag{5}$$

can be obtained by the QZ algorithm. $\Lambda$ is a matrix in Jordan form of the generalized eigenvalues $\lambda_i$ and $Y$ a matrix of all generalized eigenvectors, ordered column wise. In practice $\Lambda$ reduces to a diagonal matrix. The general solution of the homogeneous linear system is therefore given by $X(t)=Y\exp(\Lambda\,t)$. Hence the column vectors of $X$ span the solution space of the linearized DAE.

$X(t)$ is referred to as the fundamental or Wronski matrix. The state transition matrix $\Phi(t)$ is therefore $\Phi(t)=X(t)X^{-1}(0)=Y\exp(\Lambda\,t)\;Y^{-1}$. For a given initial condition $x_0\neq\bar{x}_0$ the solution of the linear system is $\Phi(t)\,x_0$ for any $t$. In what follows $x_0(t_1)\not\equiv\bar{x}_0$, i.e. the unstable operating point shall not be the initial value. Moreover let $\Delta x_0=\|x_0-\bar{x}_0\|_2$. Due to the instability of the operating point there exists at least one pair of conjugate complex eigenvalues $\lambda,\lambda^*$ with corresponding conjugate complex eigenvectors $y,y^*$, $\|y\|_2=1$ with positive real part. Therefore it makes sense to initialize the partial DAE by

(i) $\omega_0(0)=\mathrm{Im}\{\lambda\}$ and

(ii) $x_0(t_1)=\bar{x}_0+\left(\frac{\Delta x_0}{\sqrt{4\pi}}\right)y\exp(j\,t_1)+\left(\frac{\Delta x_0}{\sqrt{4\pi}}\right)y^*\exp(-j\,t_1)$, $j=\sqrt{-1}$.

### B. Estimate of the instantaneous frequency

The multirate PDE assumes a slowly varying or lowpass envelope for making the method efficient. A key issue is therefore the knowledge of the instantaneous frequency $\omega$. For driven systems it is given a priori. For autonomous systems however it must be estimated from the calculated waveforms. Note that (2) is under-determined. The instantaneous frequency is well defined via the Hilbert transformation. Let

$$x_{\mathrm{RF}}(t):\quad \mathbb{R}\to\mathbb{R}$$

be the real radio frequency (RF) or bandpass signal and

$$x(t):\quad \mathbb{R}\to\mathbb{C}$$

the complex baseband signal. Employing the Hilbert transformation there exist a unique relation between the two signals for a given modulation frequency $f_0$. Given the impulse response $h(t)=\frac{1}{\pi t}$, the Hilbert transformation of $x_{\mathrm{RF}}(t)$ is defined by the convolution

$$\mathcal{H}\{x_{\mathrm{RF}}(t)\}:=x_{\mathrm{RF}}(t)*\frac{1}{\pi t}$$

The analytical signal, which is characterized by a solely positive spectrum $X(f)\equiv0\Leftrightarrow f<0$, reads

$$x^+_{\mathrm{RF}}(t)=x_{\mathrm{RF}}(t)+j\mathcal{H}\{x_{\mathrm{RF}}(t)\}$$

Given a modulation frequency $f_0$ which is in most practical cases the center frequency of the RF signal, the relation between the analytical and bandpass signals

$$x^+_{\mathrm{RF}}(t)=\sqrt{2}\,x(t)\,\mathrm{e}^{j2\pi f_0 t}$$

holds[1]. Therefore the complex baseband signal is obtained by

$$x(t)=\frac{1}{\sqrt{2}}\,x^+_{\mathrm{RF}}(t)\cdot\mathrm{e}^{-j2\pi f_0 t}=\frac{1}{\sqrt{2}}\,(x_{\mathrm{RF}}(t)+j\mathcal{H}\{x_{\mathrm{RF}}(t)\})\cdot\mathrm{e}^{-j2\pi f_0 t}$$

The bandpass signal is therefore obtained by

$$x_{\mathrm{RF}}(t)=\sqrt{2}\,\mathrm{Re}\left\{x(t)\cdot\mathrm{e}^{j2\pi f_0 t}\right\}=\sqrt{2}\,|x(t)|\cdot\cos(2\pi f_0\,t+\Theta(t))$$

[1]The factor $\sqrt{2}$ ensures that the RF and baseband signals have identical power.

with instantaneous amplitude $|x(t)|$ and instantaneous phase $\Theta(t)$. The instantaneous frequency is therefore well defined by

$$f(t) = f_0 + \frac{1}{2\pi}\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}$$

Specifically, the instantaneous frequency is independent of the choice of the modulation frequency $f_0$.

The Hilbert transformation performs a 90° phase shift on the RF waveform. There exist a variety of analog and digital circuits approximating the impulse response of the Hilbert transformation within some frequency range of interest. For a numerical treatment the evaluation is too expensive and hence inappropriate in circuit simulation. We resort therefore to alternative techniques with small numerical complexity. In [4] two heuristic methods have been tested:

1) The linearized system of the discretized PDE (2), employing Newton's method is rank one deficient. It is solved in a least squares sense. A numerically efficient way for this calculation is also presented in [4]. This method however is not invariant w.r.t. scaling. The instantaneous frequency is therefore scaled to its initial estimate $\omega_0(0)$ discussed above.
2) In periodic steady state simulation any time shifted solution solves the DAE as well. It is well known that a unique solution can be obtained by fixing the phase. This gives rise to the following method. The phase of one specific waveform, say $\hat{x}_1(\tau, t_1)$ is fixed to the phase of the inital value at $\tau = 0$, i.e.

$$\frac{\langle \hat{x}_1(\tau,\cdot), \cos\rangle}{\langle \hat{x}_1(\tau,\cdot), \sin\rangle} = \frac{\frac{1}{2\pi}\int_0^{2\pi} \hat{x}_1(\tau,t_1)\cos(t_1)\,\mathrm{d}t_1}{\frac{1}{2\pi}\int_0^{2\pi} \hat{x}_1(\tau,t_1)\sin(t_1)\,\mathrm{d}t_1} = \frac{\langle \hat{x}_1(0,\cdot), \cos\rangle}{\langle \hat{x}_1(0,\cdot), \sin\rangle}$$

The inner products can be calculated by quadrature.

An alternative, which is invariant of scaling, solves the PDE (2) with a constraint [14]: the locally optimal frequency is obtained as follows: Let $\hat{x}(\tau_k, t_1)$, $\hat{x}(\tau_{k-1}, t_1)$ be the waveforms after semi-discretization of (2) employing the method of lines or Rothe's method. The optimal instantaneous frequency $\omega_k = \omega(\tau_k)$ w.r.t. a smooth envelope is calculated for the interval $(\tau_k, \tau_{k-1})$ by minimizing

$$\int_0^{2\pi} |\hat{x}(\tau_k, t_1) - \hat{x}(\tau_{k-1}, t_1)|^2 \,\mathrm{d}t \tag{6}$$

The integral is evaluated numerically as shown next. Let $F(X, \omega)$ be the nonlinear algebraic system from the discretized PDE (2) using one of the discretization methods discussed in the next Section with $2K+1$ disjoint grid points on the interval $[0, 2\pi)$. Then $F : \mathbb{R}^{(2K+1)N} \times \mathbb{R} \to \mathbb{R}^{(2K+1)N}$. Moreover let $A = D_X F(X, \omega)$ be the Jacobian w.r.t. the vector $X$ and $z = D_\omega F(X, \omega)$. Solving $F(X, \omega) = 0$ similar to Newton's method results in the linear equation

$$A\, d_X + d_\omega\, z = b$$

with $b = -F(X, \omega)$ subject to the constraint (6). Here the count $\ell$ of the Newton iteration is dropped for convenience. In practical applications $A$ is regular and can be inverted. A solution which satisfies the minimization constraint for $(X_k, \omega_k)$ is

$$d_X = \tilde{b} - d_\omega \tilde{z}$$

where $\tilde{b} = A^{-1} b$, $\tilde{z} = A^{-1} z$ and

$$d_\omega = \frac{\tilde{z}^T \left(X_k^{(\ell)} - X_{k-1} - \tilde{b}\right)}{\tilde{z}^T \tilde{z}}$$

Accordingly, the Newton updates are obtained by

$$X_k^{(\ell+1)} = X_k^{(\ell)} + d_X$$
$$\omega_k^{(\ell+1)} = \omega_k^{(\ell)} + d_\omega$$

The main computational work lies in calculating the $LU$ decomposition of $A$. The overhead for evaluating $d_\omega$ is therefore negligible.

## III. Discretization of the multirate PDE

First, a semi discretization by the method of lines is performed along the $t_1$ axis, resulting in a system $\tilde{f}(\hat{x}, \dot{\hat{x}}) = 0$ in $\tau$ with initial values $\hat{x}(0, t_1)$. This system can be solved with standard single or multistep methods until the steady state is achieved. The instantaneous frequency is obtained by the method described above (6).

A severe problem in simulating oscillators with high Q factor however is the discretization along the $t_1$ axis with its periodic boundary condition. Standard methods such as the BDF methods fail to calculate the solution correctly due to a numerical dissipation or loss of energy. Alternatives besides HB are trigonometric splines [16] or a novel class of multistep methods [2] without numerical dissipation for the spectral components at the fundamental frequency and the first harmonics. We employ these methods since they are closely related to the BDF formulas and easy to implement, exhibiting the same sparsity structure as BDF methods. We choose on the interval $[0, 2\pi)$ a set of $2K+1$ mutually disjoint grid points[2] $t_l$, $l = 0, \ldots, 2K$ in ascending order and $t_0 = 0$. Moreover the sparsity structure of the BDF-$p$ methods shall be preserved, i.e. $\nabla = (a_{ij})$ with $a_{ij} = 0 \;\forall\, j \not\equiv i-p, \ldots, i \pmod{2K+1}$, $i = 1, \ldots, 2K+1$, where $\nabla$ is the numerical differentiation operator. For $|k| \le L \le K$ the harmonic waveforms

$$x_k = [\exp(-jkt_0), \ldots, \exp(jkt_l), \ldots, \exp(jkt_{2K})]^T \tag{7}$$

shall be eigenvectors of $\nabla$. Moreover the corresponding eigenvalues of $\nabla$ shall be $\lambda_k = jk$, $\forall |k| \le L$. That is, the derivative is exact if the waveform is a harmonic polynomial with harmonics $|k| \le L$. For the special case $L = K$ one obtains the Sample Balance method which is equivalent to HB but avoids the transformation into the frequency domain. For $L \ll K$ one gets a highly sparse operator $\nabla$. For an equidistant discretization, i.e. $t_{l+1} - t_l = \Delta t$, the operator $\nabla$ is circulant and all eigenvectors are harmonics given by (7). Employing the notation $[i-j] := i - l \pmod{2K+1}$ the $i$-th row of the differentiation operator reads as

[2] $2K+1$ grid points are chosen for a simple comparison with HB: the frequency indices range from $-K \le k \le K$.

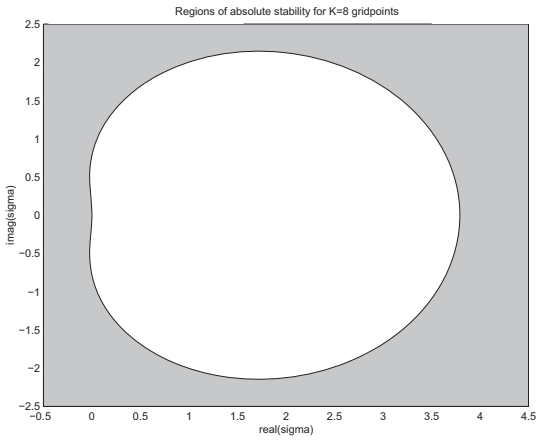



Fig. 1. Regions of absolute stability of the modified BDF scheme $p = 2$.

$$\lambda_k = \sum_{l=0}^{p} a_{i,[i-l]} \exp\left(jk(t_i - t_{i,[i-l]})\right) = jk,\ a_{i,[i-l]} = 0$$
$$\forall\, i = 1,\ldots,N, \quad |k| \le L$$

One obtains for $p+1$ coefficients $2L+1$ equations, hence there exists a solution if $2L \ge p$. The eigenvalues of $\nabla$ are purely imaginary for low frequencies. Hence the displacement current and the voltages are 90° out of phase. Therefore no energy is spuriously dissipated in capacitors/inductors which reflects the correct physical behavior. In contrary the BDF methods lead to eigenvalues with a positive real part and therefore a numerical dissipation of energy. In [2] it has been shown that this discretization formula is consistent and moreover in the limit for vanishing time steps $\Delta t_l$ one obtains the same coefficients as for Gear's BDF formulas.

The method is exemplified by the modified BDF-2 method for an equidistant grid. Let $z = \frac{2\pi}{2K+1}$. The trigonometric BDF-2 method is given by

$$\dot{x} \approx \frac{1}{\Delta t} \sum_{i=0}^{2} a_i\, x_{n+1-i}$$

where the coefficients are the solution of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \cos z & \cos 2z \\ 0 & -\sin z & -\sin 2z \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

namely

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \left( -z\frac{\cos z - \cos 2z}{\sin 2z - 2\sin z} \quad z\frac{\sin z}{\cos z - 1} \quad z\frac{1}{2\sin z} \right)^T$$

It is easy to prove that in the limit for $K \to \infty$ the coefficients coincide with the BDF method employing rational polynomials. Furthermore one can prove that if the signal is a sinusoidal waveform with angular frequency $\omega = \frac{2\pi}{(2K+1)\Delta t}$ the signal and its derivative are 90° out of phase, hence no spurious dissipation of energy occurs in energy conserving devices such as inductors and capacitors.

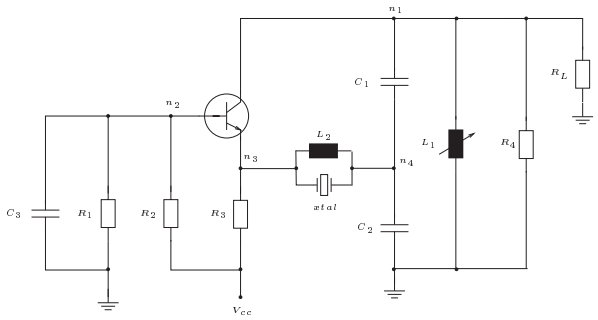



Fig. 2. Schematic of the Grounded base quartz crystal oscillator.

It shall be noted that the trigonometric BDF-2 method is not A-stable: the locus curve of the region of absolute stability of the (trigonometric) BDF methods for the test initial value problem (IVP) $\dot{x} = \lambda x$, $x(0) = 1$ is given by

$$\sigma(\Theta) = \sum_{i=0}^{p} a_i \exp(-ji\Theta), \quad 0 \le \Theta < 2\pi \tag{8}$$

where $\sigma = \Delta T\, \lambda$. The IVP exhibits the oscillation frequency $\omega_0 = \mathrm{Im}\{\lambda\}$ As for the modified (trigonometric) BDF methods the $a_i$ depend on $K$ and indirectly on the estimate of the instantaneous frequency $\omega$, so the locus curve. For $K$ grid points per oscillation period the crossing of the locus curve with the imaginary axis is $\sigma = \pm\frac{2\pi}{K}$. Fig. 1 shows exemplarily one locus curve. For weakly damped systems $\mathrm{Re}\{\sigma\} \to 0$ and a too large estimate of the oscillation frequency one observes a small region of instability:

$$-\frac{2\pi}{K}\frac{\omega_0}{\omega} \le \mathrm{Im}\{\sigma\} \le \frac{2\pi}{K}\frac{\omega_0}{\omega}$$

with estimate $\omega$. For the envelope method the weak instability is irrelevant because a boundary value problem is solved. For transient analysis the damping caused by the circuit's nonlinearities is in practice much larger.

## IV. Simulation results

A Colpitts, Pierce, Clapp and Grounded Base quartz crystal oscillator are investigated employing the circuit simulator Sframe [21]. The Fig. 2 depicts exemplarily the Grounded Base oscillator schematic. Table I depicts the calculated frequencies at steady state. The results obtained for HB and the modified BDF-2 are in excellent agreement whereas the solution for the standard BDF-2 differs slightly. Moreover standard BDF-2 fails to calculate the limit cycle for the Grounded Base oscillator. Instead the physically unstable operating point is obtained in the limit. Fig. 3 shows the envelope simulation: the oscillation dies out due to the positive real parts of the eigenvalues of the differentiation operator. The Fig. 4 and Fig. 5 show the multirate solution for the output of the 3 MHz Colpitts and 20 MHz Clapp oscillator, respectively. The envelope is smooth which confirms that the estimate of the instantaneous frequency is nearly optimal. Otherwise undulations of the envelope would be observed which requires smaller time steps and larger run time.

TABLE I

THE NUMERICALLY CALCULATED FREQUENCIES OF THE LIMIT CYCLES OF FOR SEVERAL DIFFERENCE SCHEMES AND HARMONIC BALANCE FOR 128 KNOTS. THE FREE RUNNING FREQUENCIES ARE IN $\frac{f_0}{\text{MHz}}$.

| Circuit | HB | BDF-2 | mod. BDF-2 |
|---|---|---|---|
| Colpitts I | 3.000107 | 2.997734 | 3.000106 |
| Colpitts II | 20.002824 | 19.987037 | 20.002824 |
| Pierce | 20.009107 | 19.993205 | 20.009107 |
| Clapp | 20.002316 | 19.986542 | 20.002316 |
| Grd. Base | 50.000682 | - | 50.000682 |

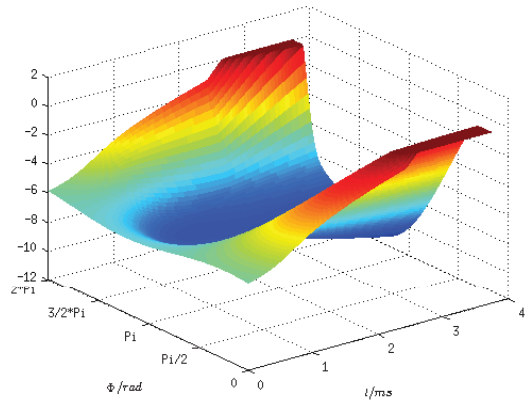



Fig. 4. Colpitts oscillator envelope following simulation by modified BDF-2.

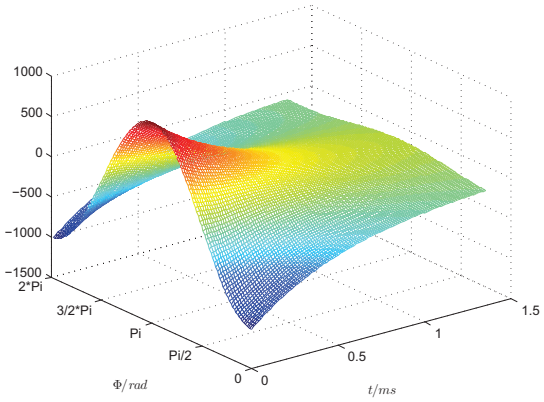



Fig. 3. Grounded Base quartz crystal oscillator envelope following simulation by BDF-2.

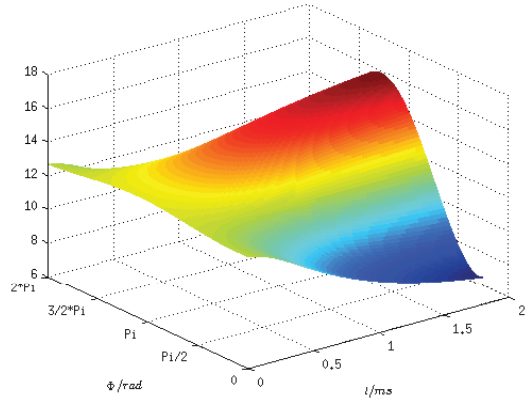


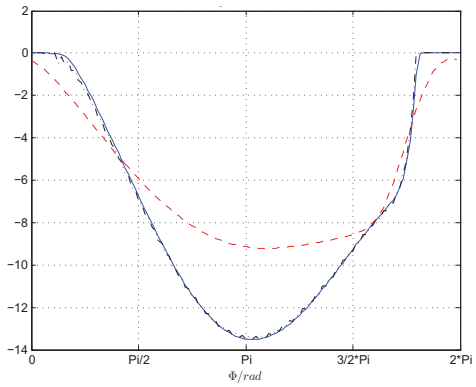

Fig. 5. Clapp oscillator envelope following simulation by modified BDF-2.

The Fig. 7 shows the numerically calculated steady states for a selected waveform employing (trigonometric) BDF-2 and HB. One can see the effect of numerical dissipation of energy caused by BDF-2: the amplitude of the waveform is much smaller compared with the competing methods. HB and modified BDF-2 are generally in an excellent agreement. However when one considers the cut out in Fig. 7(b) a spurious ringing is observed for HB whereas modified BDF-2 shows the expected smooth signal behavior. The frequency of the ringing depends on the number of grid points and cannot have a physical origin. The reason for the spurious ringing is still not clearly understood. It may be caused by the fact that trigonometric basis functions have no compact support. Moreover, the basis functions are not causal which is obviously nonphysical. Finally, the numerical differentiation operator of HB has purely imaginary eigenvalues which means that high frequency numerical artifacts are not damped.

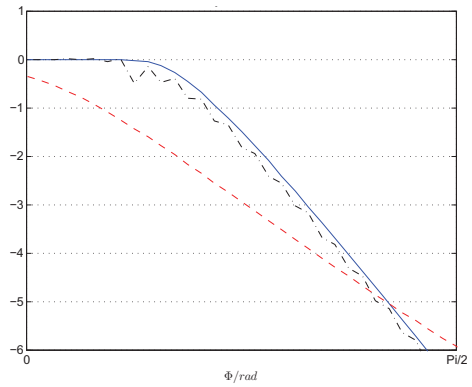
The Fig. 6 shows as second example the steady state waveforms calculated with modified BDF and HB. One can observe again the ringing of the HB solution due to the non-compactness of the trigonometric polynomials.

## V. CONCLUSION

This paper addressed the topics: 1) the calculation of the initial transient response of oscillators, 2) the estimation of the instantaneous frequency and 3) a multistep method with low numerical dissipation of energy. The bottleneck 1) caused by Nyquist's theorem has been overcome employing the multirate PDE technique. For this method however the optimal choice of the instantaneous frequency is crucial. A method is presented based on minimizing a functional in a quadratic norm circumventing the burdensome calculation of the Hilbert transform. Because multistep integration formulas such as the well known BDF methods lead to numerical damping, they are not suited for oscillators with a high quality factor. In contrary, Harmonic Balance may lead to spurious ringing which is a numerical artifact and has no physical origin. Trigonometric multistep methods are investigated which are exact if the waveforms are well approximated locally by a trigonometric polynomial of low order. These methods diminish both the unwanted effect of numerical damping and spurious ringing for all quartz crystal oscillators under test.




(a) Numerically calculated steady state solution.




(b) Numerically calculated steady state solution (cut out).

Fig. 7. Comparison of the numerically obtained limit cycle employing the second order BDF method (dashed), modified BDF method (solid) versus Harmonic Balance (dash-dot), for a mesh of 128 grid points per period for a 3 MHz Colpitts oscillator.

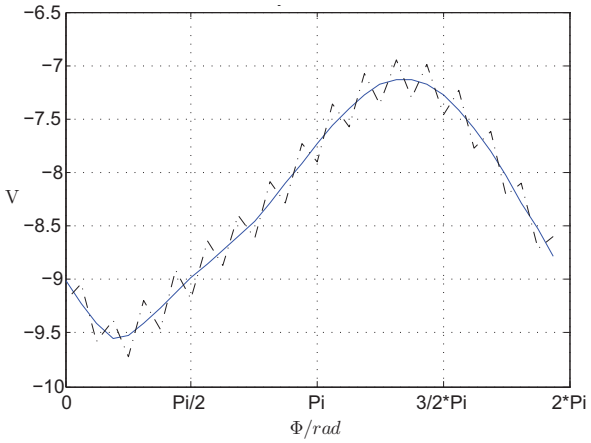



Fig. 6. Comparison of the steady state solution calculated by modified BDF (solid) and HB (dashed) for the Grounded-Base oscillator.

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