

Model-Order Reduction Based on PRONY's Method

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Abstract

A new model-order reduction technique for linear dynamic systems is presented. The idea behind this technique is to transform the dynamic system function from the s -domain into the z -domain via the bilinear transformation, then use Prony's [1] least-squares approximation method instead of the commonly employed Padé approximation method, and finally transform the reduced system back into the s -domain using the inverse bilinear transformation. Simulation results for large practical systems show that this technique based on Prony's method gives much higher accuracy than the traditional Padé method, and results in lower-order approximations without significant increase in simulation time.

1. Introduction

The computation of equivalent linear system models of large linear dynamic systems is a topic of considerable practical interest. This interest is motivated by the reduced complexity obtained by reducing the large linear subnetwork in a linear (or nonlinear) network. Ideally, linear analysis on these linear subnetworks is performed by first computing a state-space model, followed by the application of a suitable analysis method. However, the applicability of this method is limited since typical dynamic systems are represented by very large state matrices that require specialized large-scale eigen-analysis programs and computer resources. To avoid this practical limitation, model-order reduction methods are widely used in the solution of such systems. The basic idea behind model-order reduction is to replace the original system equations for the large linear network by an equivalent system with a much smaller state-space dimension. In particular, the identified reduced-order model frequency characteristics must approximate those of the full-order model.

In general, from approximation theory, there are four major categories of approximation methods that one can use depending on the overall accuracy, efficiency, and reliability desired [2]. The min-max methods rely on nonlinear optimization techniques which make them inefficient but highly accurate. Series expansion based methods are computationally efficient but may provide inaccurate results. Interpolation methods, on the other hand, are computationally effi-

cient methods which agree exactly with the original system on sample points but, in general, give unpredictable accuracy at other points [3]. Finally, least-squares methods combine the accuracy feature of the min-max methods and the efficiency of the series expansion and interpolation methods by controlling the error between the original system function and the approximate function over all points and not just where the maximum error occurs (min-max) or at the sample data points (series expansion and interpolation).

The s -domain Padé approximation [4] is a combination of series expansion and interpolation methods that has been used in the asymptotic waveform evaluation (AWE) [5] algorithm in order to extract the dominant poles and residues of the system. Other Padé techniques based on Krylov-subspace methods — such as Padé via Lanczos (PVL) [6] and Arnoldi-based model-order reduction [7, 8] — provide *efficient* estimation of the original system response. However, the *accuracy* of these methods is limited by the order of the Padé approximation (i.e. the number of moments matched). The problem with the Padé approximation is that it equates the approximating transfer function to the original system function to obtain as many equations as there are unknowns [3,4]. This equating technique is the major limitation in the Padé approximation because the resulting transfer function must contain a large number of poles and zeros in order for it to be sufficiently close to the original system.

In this paper, we present a novel model-order reduction technique for obtaining a reduced-order transfer function approximation. The technique is based on three steps. First, the original large system transfer function is transformed from the s -domain into the z -domain via the bilinear transformation. It is well known that the bilinear transformation always preserves system stability, and can always be made to preserve the system frequency response characteristics for a specified frequency range [9, 10]. Second, we propose to apply Prony's least-squares approximation method to reduce the order of the *transformed* system function. Third, the reduced-order system is transformed back to the s -domain using the inverse bilinear transformation. We show that Prony's approximation method performs much better than the traditional Padé approximation method. Furthermore, the derivation of this model-order reduction technique enables the different Krylov-subspace algorithms [6–8], traditionally

applied for the Padé approximation, to be used with Prony's method to obtain efficient and accurate moment approximations.

The remainder of the paper is organized as follows. In Section 2, we describe the s -to- z system transformation together with moment computation techniques. We also describe the connection of the different Krylov-subspace algorithms in this new representation. In Section 3, we explain Prony's least-squares approximation method and compare it to the traditional Padé approximation. We corroborate the derived results in Section 4 with simulations of large practical systems and show the accuracy of our technique. Finally, Section 5 provides some concluding remarks.

2 s -to- z System Transformation

In this section, we briefly describe the bilinear transformation and then apply this transformation to represent a generalized linear, time-invariant (LTI) system in the z -domain. We then describe three methods for computing the moments of the z -domain system function.

2.1 The bilinear transformation

The bilinear transformation is a nonlinear mapping $\beta : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$\beta : s \mapsto z = \frac{\alpha + s}{\alpha - s}, \quad (1)$$

where $\alpha \in \mathbb{R}$ is a constant equal to twice the sampling rate. This mapping has the property that it transforms the $j\Omega$ -axis in the s -plane onto the unit circle ($z = e^{j\omega}$) in the z -plane [9]. Moreover, the left-half s -plane ($\text{Re}(s) < 0$) is mapped inside the unit circle in the z -plane and the right-half s -plane ($\text{Im}(s) > 0$) is mapped outside the unit circle in the z -plane, thus preserving system stability. The inverse bilinear transformation is given by

$$\beta^{-1} : z \mapsto s = \alpha \frac{z-1}{z+1}. \quad (2)$$

The main advantage of using the bilinear transformation over other transformations such as the impulse- and step-invariant transformations, is that it preserves the magnitude characteristics of the transfer function. This follows from the fact that the parameter α provides a one degree of freedom that can be used to make the frequency response characteristics of the z -domain system function $\mathbf{H}_d(z)$ approximate those of the s -domain system function $\mathbf{H}_c(s)$ [10].

2.2 z -domain system representation

A generalized LTI system of equations can be used to model a dynamic system as follows:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + bu(t), \\ y(t) &= c^T x(t) + du(t), \end{aligned} \quad (3)$$

where $x \in \mathbb{R}^n$ represents the state variables of the system, $y \in \mathbb{R}^p$ are the p outputs of the system defined using $c \in \mathbb{R}^{n \times p}$, E

and $A \in \mathbb{R}^{n \times n}$, and b and $d \in \mathbb{R}^{n \times m}$ represent excitations from m independent sources. It can be shown that the resulting system transfer function of (3) is given by

$$\mathbf{H}_c(s) = c^T (sE - A)^{-1} b. \quad (4)$$

Applying the bilinear transformation in (1) to (4) for an appropriately chosen α to preserve the magnitude characteristics, the z -domain transfer function becomes

$$\begin{aligned} \mathbf{H}_d(z) &\triangleq \mathbf{H}_c(s) \Big|_{s=\alpha \frac{1-z^{-1}}{1+z^{-1}}} \\ &= c^T (1+z^{-1})(I - z^{-1}M)^{-1} r, \end{aligned} \quad (5)$$

where $M = (\alpha E - A)^{-1}(\alpha E + A)$ and $r = (\alpha E - A)^{-1}b$. In order to invert the term $(I - z^{-1}M)$ in (5), we need to diagonalize M , which is numerically expensive and impractical to perform for typical large dynamic systems. Instead, one resorts to using Neumann's expansion [11]:

$$(I - z^{-1}M)^{-1} = I + z^{-1}M + z^{-2}M^2 + \dots \quad (6)$$

Substituting (6) in (5), the z -domain transfer function $\mathbf{H}_d(z)$ becomes

$$\mathbf{H}_d(z) = (1 + z^{-1}) \sum_{i=0}^{\infty} m_{c,i} z^{-i}, \quad (7)$$

where $m_{c,i} = c^T M^i r$ are the s -domain moments of $\mathbf{H}_c(s)$. $\mathbf{H}_d(z)$ can also be written in a more convenient form by defining the z -domain moments

$$m_{d,i} = m_{c,i-1} + m_{c,i}, \quad (8)$$

as

$$\mathbf{H}_d(z) = \sum_{i=0}^{\infty} m_{d,i} z^{-i}. \quad (9)$$

2.3 Computing the moments

It is usually sufficient to compute the first $w+1$ moments in (7) or (9), where w depends on the choice of the approximation algorithm used in the second step of the proposed model-reduction technique. In AWE [5], the moments $m_{c,i}$ are explicitly computed by first recursively solving the linear system of equations

$$(\alpha E - A)u_i = -Eu_{i-1}, \quad i = 1, 2, \dots, w, \quad (10)$$

for u_i with $u_0 = r$, and then $m_{c,i} = c^T u_i$. However, due to finite machine precision, this approach is numerically ill-conditioned; see [6]. A better alternative is to use Krylov-subspace methods which enable a more stable computation of the moments. For instance, in the Lanczos algorithm, these moments are obtained by

$$m_{c,i} = l^T \text{re}_1^T T_w^i e_1, \quad i = 0, 1, \dots, w, \quad (11)$$

where e_1 is the first unit vector in \mathbb{R}^w and T_w is a tridiagonal matrix; see, e.g., [6] for further details. Similarly, in the Arnoldi process the $m_{c,i}$ moments are given by

$$m_{c,i} = \|r\|_2 l^T V_w H_w^i e_1, \quad i = 0, 1, \dots, w, \quad (12)$$

where $V_w \in \mathbb{R}^{n \times w}$, and H_w is a $w \times w$ upper Hessenberg matrix whose scalar entries are generated by the Arnoldi algorithm; see [12] for details. Finally, the z -domain moments are easily computed from the $m_{c,i}$ moments using (8).

3 Reduced-Order Modeling

The model-order reduction problem can now be stated as follows: Given the *transformed* z -domain system response as described by equation (9), find an approximate z -domain transfer function of the form

$$\tilde{\mathbf{H}}_d(z) = \frac{\sum_{k=0}^{q-1} b_k z^{-k}}{1 + \sum_{k=1}^q a_k z^{-k}} \equiv \sum_{i=0}^w \tilde{m}_{d,i} z^{-i}, \quad (13)$$

whose response $\tilde{m}_{d,i}$ accurately approximate the response $m_{d,i}$ of $\mathbf{H}_d(z)$. Once this approximate system is identified, it is transformed back into the s -domain using the inverse bilinear transformation in (2) in order to obtain the reduced-order s -domain transfer function.

The rational representation of the approximate transfer function given in equation (13) has $2q$ unknowns, namely, the coefficients $\{a_k\}_{k=1}^q$ and $\{b_k\}_{k=0}^{q-1}$. Existing techniques employ the Padé approximation to determine these coefficients [5–7]. The drawback of this technique is that it equates the $w+1$ terms of $\tilde{\mathbf{H}}_d(z)$ of (13) to the first $w+1$ moments of $\mathbf{H}_d(z)$ in (9) to get as many equations as there are unknowns. This technique, in turn, requires using a large-order rational approximate function to achieve acceptable accuracy. To solve this problem, we propose using Prony's least-squares method which determines the moments of $\tilde{\mathbf{H}}_d(z)$ over a large number of moments but using a low-order rational approximate function.

3.1 Padé approximation

We will first briefly present the z -domain Padé approximation which is similar to the commonly used s -domain Padé approximation [5, 6] in order to compare with our proposed method.

In this approximation, the parameter w is set to $2q-1$ and the responses $\tilde{m}_{d,i}$ and $m_{d,i}$ of the two transfer functions are equated for $0 \leq i \leq 2q-1$, in order to obtain as many equations as there are unknowns. This results in the following system of linear equations

$$\tilde{m}_{d,i} = m_{d,i} = \begin{cases} -\sum_{k=1}^q a_k m_{d,i-k} + b_i, & 0 \leq i \leq q-1, \quad (14a) \\ -\sum_{k=1}^q a_k m_{d,i-k}, & q \leq i \leq 2q-1, \quad (14b) \end{cases}$$

where it is assumed that moments with negative indices are zero. The coefficients $\{a_k\}$ are first obtained by solving q equations in (14b), which are then used to determine the coefficients $\{b_k\}$ in (14a).

Thus, the Padé approximation method results in a perfect match between the $\tilde{m}_{d,i}$ and the original $m_{d,i}$ for the first $2q$ values of the impulse response. However, for $i \geq 2q$ there

is no bound on the error between the two responses. This is the major limitation with the Padé approximation, and hence the resulting transfer function must contain a large number of poles and zeros in order for its impulse response to be sufficiently close to the response of the original transfer function. Furthermore, the Padé approximation results in a perfect match with the original moments $m_{d,i}$ only when the original system is rational and we have prior knowledge of its number of poles and zeros. However, this is not usually the case since we only know the impulse response data as given in (7).

3.2 Prony's approximation

The main problem with the Padé approximation is that it sets $w = 2q-1$ in order to get as many equations as there are unknowns. In Prony's least-squares method [1, 13], this constraint is removed and w can be a large number ($\gg 2q-1$) without necessarily resulting in a large-order rational function. This method tries to minimize the linear least-squares error between the two responses

$$\varepsilon = \sum_{i=0}^w |m_{d,i} - \tilde{m}_{d,i}|^2, \quad (15)$$

with respect to the coefficients $\{a_k\}$ of the approximate transfer function $\tilde{\mathbf{H}}_d(z)$. This linear least-squares optimization minimizes the linear prediction error, and not the original minus approximate squared response error. The idea behind the linear prediction part is that instead of equating $\tilde{m}_{d,i} = m_{d,i}$ for all i , recursively compute $\tilde{m}_{d,i}$ using a linear predictor defined by the following set of linear equations

$$\tilde{m}_{d,i} = \begin{cases} -\sum_{k=1}^q a_k m_{d,i-k} + b_i, & 0 \leq i \leq q-1, \quad (16a) \\ -\sum_{k=1}^q a_k m_{d,i-k}, & q \leq i \leq w. \quad (16b) \end{cases}$$

The coefficients $\{a_k\}$ are then chosen so as to minimize the squared-prediction-error defined by

$$\varepsilon = \sum_{i=q}^w \left| m_{d,i} + \sum_{k=1}^q a_k m_{d,i-k} \right|^2, \quad l = 1, \dots, q.$$

Setting $\partial \varepsilon / \partial a_l$ to zero, results in the following set of linear equations

$$\sum_{k=1}^q a_k \sum_{i=q}^w m_{d,i-k} m_{d,i-l} = -\sum_{i=q}^w m_{d,i} m_{d,i-l}, \quad l = 1, \dots, q.$$

This set of equations can equivalently be written as

$$\sum_{k=1}^q a_k Y_{k,l} = -y_l, \quad l = 1, \dots, q, \quad (17)$$

where, by definition,

$$Y_{k,l} = \sum_{i=q}^w m_{d,i-k} m_{d,i-l} \quad \text{and} \quad y_l = \sum_{i=q}^w m_{d,i} m_{d,i-l}.$$

Equation (17) can be used to determine the coefficients $\{a_k\}$ which are then used to determine the $\{b_k\}$ coefficients¹ using

$$b_k = \tilde{m}_{d,k} + \sum_{i=1}^q a_i m_{d,k-i}, \text{ for } k = 0, \dots, q-1. \quad (18)$$

4 Applications

In this section, we apply the proposed model-order reduction technique using Prony's method on four large practical systems and compare their resulting responses with the corresponding original system response. We also show the responses resulting from using the commonly employed Padé approximation method. The simulation times for the four examples are shown in Table 1.

4.1 Fifth-order Chebyshev filter

Consider a fifth-order Chebyshev filter with system function as given in equation (19), where $\omega_c = 2\pi \times 10^4$ rad/sec. The circuit implementation of this filter is shown in Figure 1. It consists of two second-order LCR resonators and a first-order op amp-RC circuit. Real 741 op amp circuits were used in the design. Using SPICE, we performed an AC analysis over the linear frequency range of 1 Hz to 300 MHz. The original system was of order 720. Using modified nodal analysis (MNA) stamps extracted from SPICE, we performed the two approximation procedures and the resulting system responses are plotted in Figure 2. As seen, a 6th-order Padé approximation of the resulting frequency response is a relatively poor approximation to the original response. However, Prony's method performs significantly better using only a 4th-order approximation. By increasing the order of the Padé approximation to a 7th-order approximation we finally obtain a good match with the original Chebyshev filter.

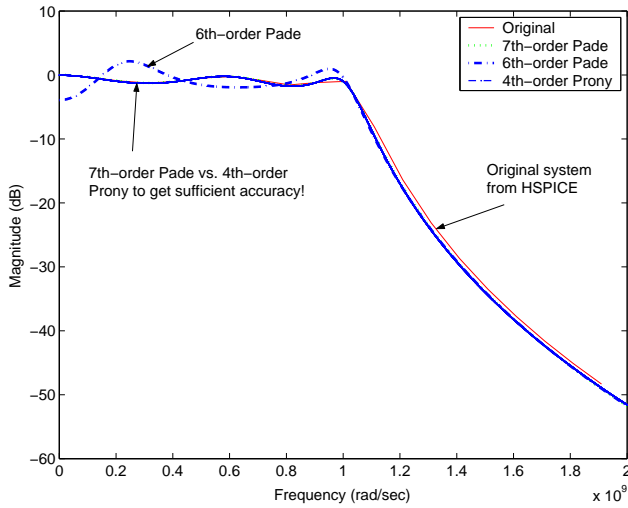


Figure 2. Chebyshev filter response.

¹The denominator coefficients can also be determined using a least squares minimization via a technique known as SHANK's method [14].

4.2 Clamped beam

The clamped beam model of [15] has 348 states. The input represents the force applied to the structure at the free end, and the output is the resulting displacement. We plot in Figure 3 the magnitude response of the original system together with that of the reduced models. The Padé method requires a system of order 54 to obtain an exact match, whereas Prony's method requires only a 40th-order approximation.

4.3 Large-order system

This example is similar in spirit to the example proposed in [16]. The system is of order 3018. The system eigenvalues are $\lambda(A) = \{-1 \pm j10, -1 \pm j700, -1 \pm j1200, -1 \pm j1400, -1 \pm j2600, -1 \pm j5800, -1 \pm j7000, -1 \pm j12000, -1 \pm j42000, -1, -2, \dots, -3000\}$. As can be seen in Figure 4, using Prony's method a 35th-order system accurately approximates the magnitude waveform of the original system. In contrast, a 35th-order Padé approximation fails to capture the magnitude response for frequencies less than 200 rad/sec.

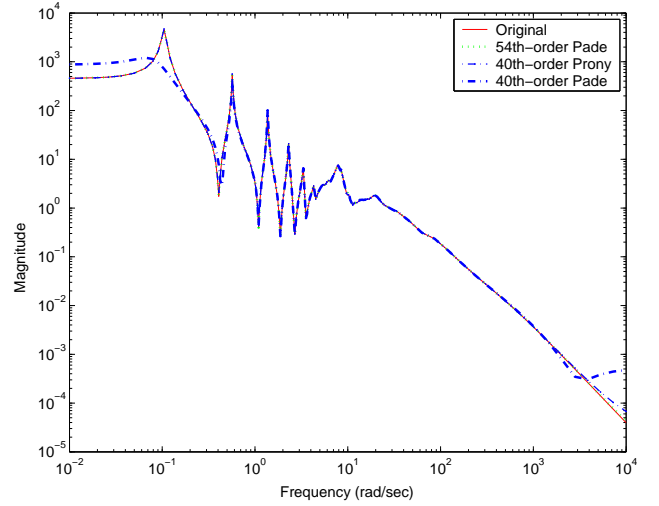


Figure 3. Clamped beam system response.

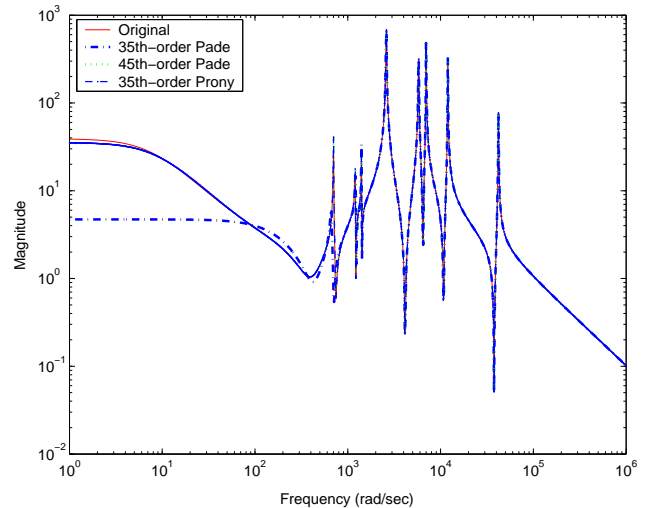


Figure 4. Large order system response.

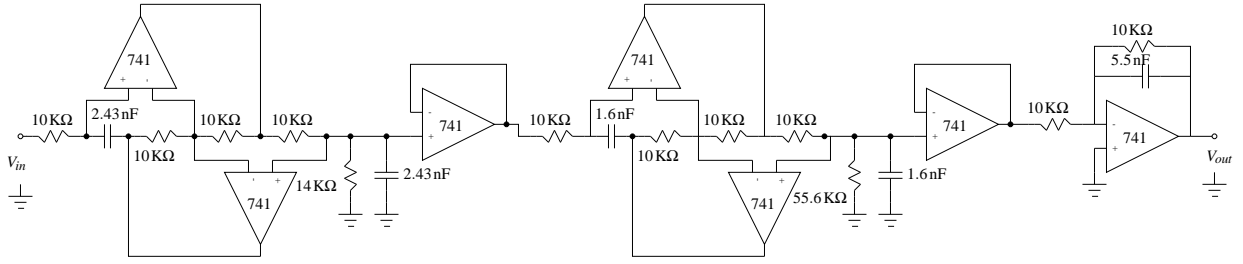


Figure 1. Fifth-order Chebyshev filter circuit schematic implemented using real 741 op amps.

$$\mathbf{H}_c(s) = \frac{\omega_c^5}{8.1408(s + 0.2895\omega_c)(s^2 + 0.4684\omega_c s + 0.4293\omega_c^2)(s^2 + 0.1789\omega_c s + 0.9883\omega_c^2)}, \quad (19)$$

4.4 International space-station

This example is a structural model of component 1r (Russian service module) of the International Space Station (ISS) [17]. We plot in Figure 5 the magnitude responses for $\mathbf{H}_{c,11}(s)$, $\mathbf{H}_{c,12}(s)$, $\mathbf{H}_{c,13}(s)$ and $\mathbf{H}_{c,21}(s)$, $\mathbf{H}_{c,22}(s)$, $\mathbf{H}_{c,23}(s)$ of the original system as well as the reduced models. As shown in the plots, Prony's method accurately models the original system with lower orders than the Padé method.

5 Conclusion

In this paper, we have presented a new model-order reduction technique for approximating large dynamic systems. We have shown through simulations of large practical systems that Prony's method can produce lower-order approximations than the Padé approximation while maintaining sufficient accuracy and without significant increase in simulation time (see Table 1).

Table 1. Comparison in simulation times.[§]

Example	Padé		Prony	
	w	time (sec)	w	time (sec)
Chebyshev filter [‡]	13	1.07	50	1.43
Clamped beam [†]	107	0.54	120	0.84
Large system [†]	89	2.27	120	3.38
ISS 1-to-1 [‡]	49	0.29	80	0.78
ISS 1-to-2 [‡]	53	0.29	80	0.85
ISS 1-to-3 [‡]	53	0.29	80	0.81
ISS 2-to-1 [‡]	83	0.77	120	1.28
ISS 2-to-2 [‡]	49	0.29	80	0.58
ISS 2-to-3 [‡]	89	0.79	120	1.20
Average Time		0.73		1.24

[§] Simulated with MATLAB on Sun Ultra 10 workstation.

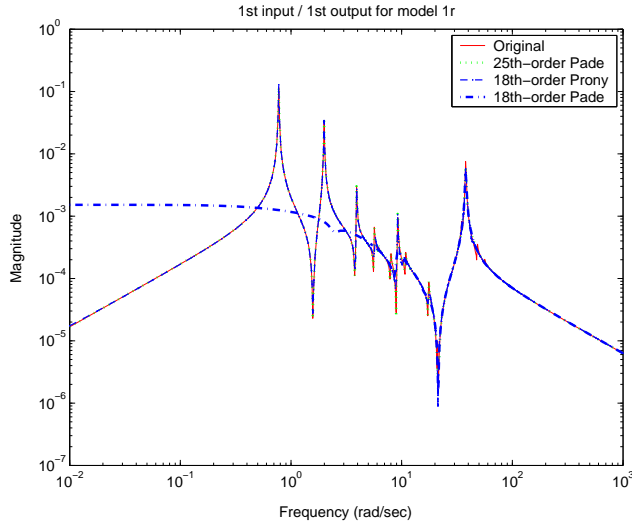
[‡] Moments computed using Arnoldi process [8].

[†] Moments computed using Lanczos process [6].

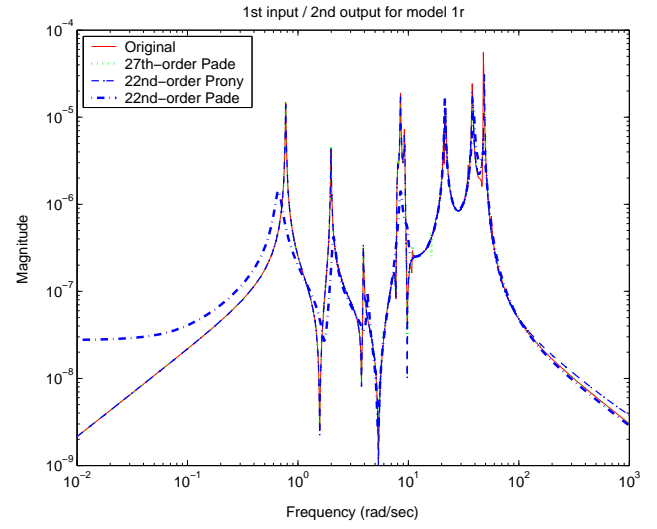
[‡] Moments computed using AWE technique [5].

References

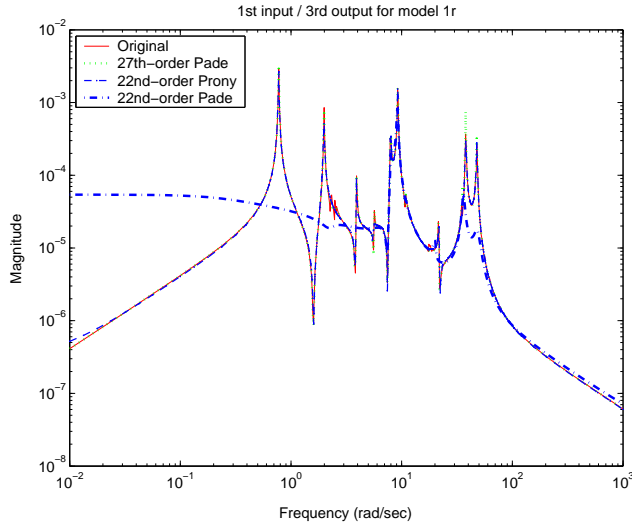
- [1] G. Baron de Prony, "Essai expérimental et analytique: Sur les lois de la dilatabilité de fluides élastiques et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures," *Jour. de L'Ecole Polytechnique*, vol. 1, pp. 24–76.
- [2] D. Kuzentsov and J. Scutt-Ainé, "Optimal transient simulation of transmission lines," *IEEE TCAS*, vol. 43, no. 2, pp. 110–121.
- [3] M. Haque, A. El-Zein, and S. Chowdhury, "A new time-domain macromodel for transient simulation of uniform/nonuniform multiconduction transmission-line interconnections," in *IEEE/ACM DAC*, 1994, pp. 628–633.
- [4] G. Baker Jr., *Essentials of Padé Approximants*, Academic Press, 1975.
- [5] L. Pillage and R. Rohrer, "Asymptotic waveform evaluation for timing analysis," *IEEE TCAD*, vol. CAD-9, pp. 352–366, Apr. 1990.
- [6] P. Feldman and R. Freund, "Efficient linear circuit analysis by Padé approximation via the Lanczos process," *IEEE TCAD*, vol. 14, pp. 639–649, May 1995.
- [7] E. Grimme, *Krylov projection methods for model reduction*, Ph.D. thesis, University of Illinois, 1997.
- [8] A. Odabasioglu, M. Celik, and L. Pileggi, "PRIMA: Passive reduced-order interconnect macromodelling algorithm," *IEEE TCAD*, pp. 645–653, Aug. 1998.
- [9] C. Chen and D. Wong, "Error bounded Padé approximation via bilinear conformal transformation," in *IEEE/ACM DAC*, 1999, pp. 7–12.
- [10] Makram M. Mansour and A. Mehrotra, "z-domain model-order reduction," Tech. Rep., CSL technical report, University of Illinois.
- [11] C. Chen, *Linear System Theory and Design*, Rinehart and Winston, New York, NY, 1984.
- [12] G. Golub and C. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, 3rd edition, 1996.
- [13] S. Marple Jr., *Digital Spectral Analysis*, Prentice-Hall, Englewood, NJ, 1987.
- [14] Makram M. Mansour and A. Mehrotra, "Reduced-order modeling based on PRONY's and SHANK's methods via the bilinear transformation," *IEEE International Symposium on Quality Electronic Design (to appear)*, Mar. 2003.
- [15] A. Antoulas, D. Sorensen, and S. Gugercin, "A survey of model reduction methods for large scale systems," in *Contemporary Mathematics*, AMS Publications, 2001, pp. 193–219.
- [16] T. Penzl, "Algorithms for model reduction of large dynamical systems," *Technical Report SFB393/99-40, Sonderforschungsbereich 393 Numerische Simulation auf massiv parallelen Rechnern*, 1999.
- [17] A. Antoulas, D. Sorensen, and N. Bedrossian, "Approximation of the international space station 1R and 12A flex models," in *IEEE Conference on Decision and Control*, Dec. 2001.



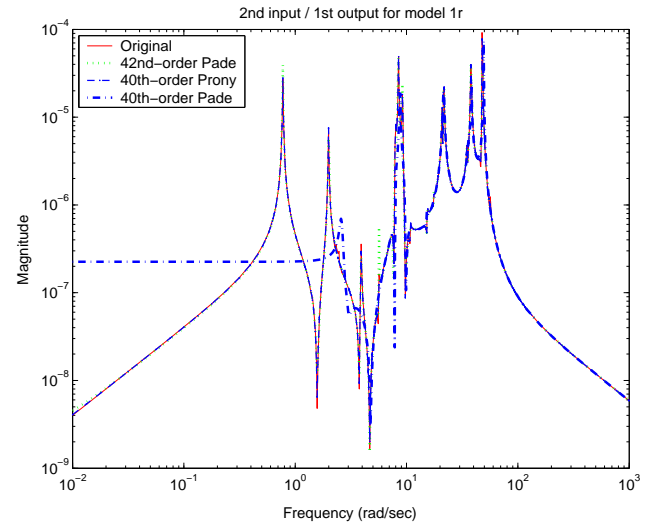
(a) $H_{c,11}(s)$



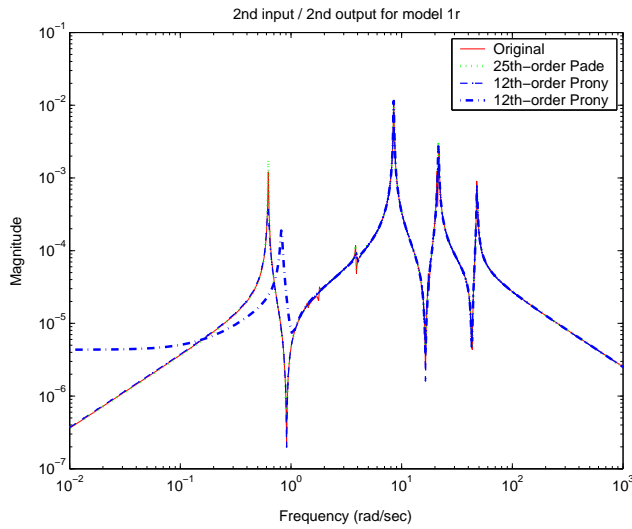
(b) $H_{c,12}(s)$



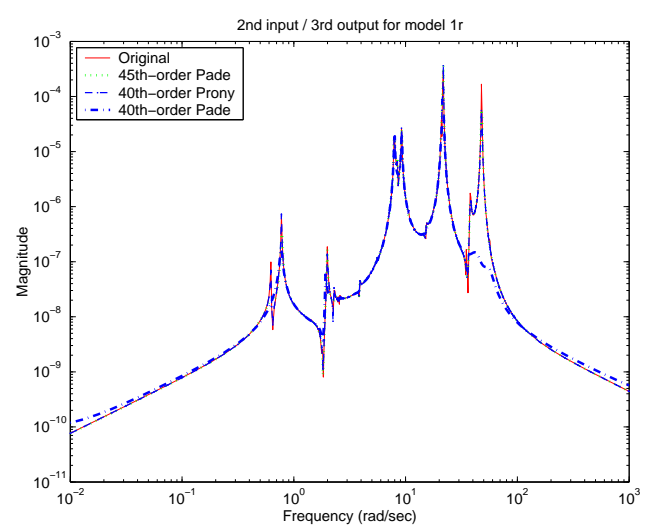
(c) $H_{c,13}(s)$



(d) $H_{c,21}(s)$



(e) $H_{c,22}(s)$



(f) $H_{c,23}(s)$

Figure 5. Magnitude response for the international space-station example.