A New Simulation Technique for Periodic Small-Signal Analysis

M M. Gourary, S. G. Rusakov, S. L. Ulyanov, M.M. Zharov
IPPM, Russian Academy of Sciences, Moscow, Russia
B. J. Mulvaney
Motorola Inc., Austin, Texas, USA

Abstract

A new numerical technique for periodic small signal analysis based on harmonic balance method is proposed. Special-purpose numerical procedures based on Krylov subspace methods are developed that reduce the computational efforts of solving linear problems under frequency sweeping. Examples are given to show the efficiency of the new algorithm for computing small signal characteristics for typical RF circuits.

1. Introduction

Periodic small-signal analysis has proven to be a useful technique in the design and verification of analog and RF circuits [1-3]. This type of analysis is used for calculation of transfer functions, noise, and distortion in periodically driven circuits.

Periodic small-signal analysis [1-6] is a two-step process:

- computation of the periodic operating point by performing a periodic steady state (PSS) analysis with the large signal stimulus. This stimulus could be, for example, a local oscillator signal or a clock signal driving the circuit.
- computation of the linear response of the periodically varying circuit to a small sinusoid of arbitrary frequency. This frequency is usually swept over the range of interest.

This is analogous to traditional ac analysis in a Spice-like simulator, where the DC operating point is computed, the circuit equations are linearized about this operating point, and the linear frequency response to small sinusoid is computed. With periodic small-signal analysis, the equations are linearized about the large signal time-varying steady state of the circuit, and as a result the response exhibits frequency conversion effects [3-5].

Note that versions of periodic small signal analysis exist for both harmonic balance (HB) and shooting methods [15]. For many classes of circuits HB is preferred over time-domain simulation for the computation of the periodic large and small signal steady state; for instance, multitone circuits with more than one large signal, high-Q circuits, and circuits with distributed models.

To provide frequency sweeping in periodic small signal analysis it is necessary to solve a number of linear problems where the matrix is a function of frequency:

$$A(\omega)x(\omega) = b(\omega) \quad (1)$$

Okumura et al. [5-6] solved the periodic small-signal problem using direct methods with Gaussian elimination for solving (1).

In recent years, Krylov subspace based iterative methods have been successfully applied to accelerate solving of large dimension problems in various RF applications. Two alternative numerical algorithms based on Krylov subspace techniques have been successfully exploited in practice: QMR [7,8] and GMRES [9,10] algorithms. In particular the special-purpose GMRES based algorithm has been developed for steady-state simulation in time domain [10]. New numerical schemes were suggested [11,12] to extend this technique for some applications.

However, the application of original GMRES algorithm for solving the set of linear problems with matrix function of frequency is typically inefficient due to linear growth of computational efforts and the special purpose computational procedures are required.

The original GMRES algorithm is not suitable for constructing the special purpose procedures for multifrequency problems. This technique includes a process of constructing an Arnoldi basis [13] that is stiffly connected with starting residual vector and matrix A with fixed entries. Thus, the obtained Arnoldi basis can not be exploited for matrices $A(\omega)$ with varying value ω.

Telichevesky et al. [3-4] applied Krylov subspace techniques to greatly improve the efficiency of the periodic small-signal analysis. To avoid the linear growth of computational complexity under frequency sweeping, they used an efficient time-domain based recycled Krylov subspace method [3-4]. However, it exploits the special form of matrix and can not be directly applied to arbitrary form of matrix $A(\omega)$ and in particular to the matrix of

1530-1591/03 $17.00 © 2003 IEEE

harmonic balance method.

In this paper we introduce a new fast frequency sweeping algorithm for periodic small signal analysis based on HB method. The new algorithm extends Krylov subspace techniques to solving of linear problems with matrix function of frequency.

This paper is organized as follows. Section 2 presents the mathematical formulation of the periodic small signal problem, and the basic numerical procedures for solving the systems (1) are presented in Section 3. Results of numerical experiments using new algorithms are given in Section 4.

2. Periodic Small Signal Harmonic Balance

Let the circuit model be represented in time domain by the standard form [14]:

$$\frac{d}{dt}q(v(t)) + i(v(t)) + u(t) + \Delta u(t) = 0 \tag{2}$$

Here *v(t), i, q* are vectors of nodal voltages, currents and charges respectively, *u(t)* is the large periodic input and $\Delta u(t)$ is a small signal sinusoidal input.

Let $v(t) = v_L(t)$ be the steady-state solution of equation (2) for $\Delta u(t) = 0$, satisfying to the periodic condition:

$$v_L(0) = v_L(T) \tag{3}$$

where *T* is the period of large signal tone.

Then the small signal steady-state response can be obtained from periodically time-varying linear system [4,5]:

$$\frac{d}{dt}(c(t)\Delta v) + g(t)\Delta v + \Delta u(t) = 0 \tag{4}$$

Here the matrices of time domain capacitances and conductances *c(t)* and *g(t)* are *T*-periodic:

$$c(t+T) = c(t) \tag{5}$$

$$g(t+T) = g(t) \tag{6}$$

For small signal sinusoidal input $\Delta u(t) = \Delta U e^{j\omega t}$ the steady-state solution of (4) has the following form:

$$\Delta v(t) = \sum_{k=-\infty}^{\infty} \Delta V(k) e^{j(\omega + k\Omega)t} \tag{7}$$

with boundary condition

$$\Delta v(t+T) = \Delta v(t) e^{j\omega T} \tag{8}$$

Here Ω, ω are fundamental frequencies of large and small signal tone respectively.

Using Fourier transform of *T*-periodic matrices *c(t)* and *g(t)*

$$g(t) = \sum_{k=-\infty}^{\infty} G(k) e^{jk\Omega t} \tag{9}$$

$$c(t) = \sum_{k=-\infty}^{\infty} C(k) e^{jk\Omega t} \tag{10}$$

and inserting (7) into (4) we can obtain the frequency-domain form of small signal equations (4):

$$\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty}\tilde{G}\Delta V(l)\right)e^{j(k\Omega+\omega)t} + \Delta U e^{j\omega t} = 0 \tag{11}$$

where

$$\tilde{G} = G(k-l) + (k\Omega + \omega)C(k-l) \tag{12}$$

Here *G(.), C(.)* are harmonics of entries of nodal conductances and capacitances matrices respectively that are calculated in large signal steady-state analysis.

Taking into account that exponent functions are linearly independent over a period *T* and considering a finite number of harmonics K_h (11) yields:

$$J\begin{bmatrix}\Delta V(-K_h)\\ \dots \\ \Delta V(0) \\ \dots \\ \Delta V(K_h)\end{bmatrix} = \begin{bmatrix}0\\ \dots \\ \Delta U \\ \dots \\ 0\end{bmatrix} \tag{13}$$

where *J* is the block Jacobi matrix with blocks

$$J_{kl} = G(k-l) + j(k\Omega+\omega)C(k-l) = J_{kl}^{HB} + \omega j C(k-l) \tag{14}$$

$k,l = -K_h,\dots,0,\dots,K_h$

The linear system (13) provides the frequency domain solution of equation (4) for a certain frequency of small signal sinusoidal input. Note that order of system (13) is $(2K_h+1)N$, where *N* is the dimension of vector *v*.

If the frequency of small signal input ω is swept we need to solve a set of linear systems (13).

Thus the small signal HB analysis includes two stages:

- large signal steady-state analysis using standard HB method (solving the problem (2,3)),
- forming and solving the set of parameterized linear problems (13) if frequency sweeping is desired for computation of small signal characteristics.

The application of iterative algorithms to solve the set of problems (13) requires efforts that are proportional to the number of frequency points. Therefore our purpose is to find such approach that allows us to exploit the solutions at previous small-signal frequencies in solving the linear problem (13) with the next small-signal frequency.

3. Computational Method for Parameterized Linear Systems

In this section we consider the problem of parameterized linear systems, i.e. linear systems with a matrix that linearly depends on scalar parameter s

$$A(s_m)x^{(m)} = b^{(m)} \qquad m = 1, \ldots, M \tag{15}$$

$$A(s) = A' + sA'' \tag{16}$$

In application to periodic small-signal analysis the parameter s is the frequency ω and M is the number of frequency points.

Paper [4] presents the Krylov subspace based recycling GCR algorithm for solving (15) that essentially exploits the special form of matrix $A(s)$, namely $A' = I$.

Here we start by introducing more general approach that can be applied to the arbitrary matrix of form (16). Then we present some improvements which reduce computational efforts and increase reliability of the algorithm.

Our purpose is to use the data accumulated in solving previous linear systems for solving new system. To do this we take into account that the principal operation required in Krylov subspace methods is matrix-vector multiplication and the number of multiplications determines the complexity of the algorithm [13]. For matrix of type (16) the matrix-vector product $A(s)y$ can be presented in the following form:

$$z(s) = A(s)y = z' + sz'' \tag{17}$$

$$\text{where } z' = A'y \qquad z'' = A''y \tag{18}$$

Therefore if we apply Krylov subspace method to solve (15) for some value of s and compute matrix-vector products $z(s)$ by calculating vectors z', z'' then we can obtain matrix-vector product $A(s)y$ for any other s by the expression (17). In comparison with explicit matrix-vector multiplication for each s this requires less computational efforts.

Now consider how to exploit previously obtained vectors for solving new system. For simplicity consider the solving of linear system

$$Ax = b \tag{19}$$

in assumption that we have obtained K matrix-vector products, i.e. such vectors y_k, z_k that

$$z_k = Ay_k \qquad k = 1, \ldots, K \tag{20}$$

We can find an approximate solution in the form

$$x = \sum_{k=1}^{K} c_k y_k \tag{21}$$

where coefficients c_k are determined by minimization of the residual norm:

$$\left\| A\left(\sum_{k=1}^{K} c_k y_k\right) - b \right\| = \left\| \sum_{k=1}^{K} c_k z_k - b \right\| \to \min_c \tag{22}$$

If vectors z_k are orthonormalized then minimization problem (22) can be solved by projecting rhs vector b onto the basis vectors z_k. For nonorthogonal vectors z_k one can apply Gram-Schmidt orthogonalization process to vectors z_k:

$$z_k = z_k - \sum_{n=1}^{k-1} (z_k, \tilde{z}_n)\tilde{z}_n \qquad \tilde{z}_k = z_k / \|z_k\| \tag{23}$$

Simultaneously the same linear transform is applied to vectors y_k:

$$y_k = y_k - \sum_{n=1}^{k-1} (z_k, \tilde{z}_n)\tilde{y}_n \qquad \tilde{y}_k = y_k / \|z_k\| \tag{24}$$

It can be easily seen that

$$\tilde{z}_k = A\tilde{y}_k \qquad k = 1, \ldots, K. \tag{25}$$

and the approximate solution of (19) is:

$$x = \sum_{k=1}^{K} \tilde{c}_k \tilde{y}_k \tag{26}$$

where $\tilde{c}_k = (\tilde{z}_k, b)$ for orthonormalized $\tilde{z}_k$.

The residual vector of the linear system (19) for the approximate solution (26) can be presented as

$$r_K = b - Ax = b - \sum_{n=1}^{K} \tilde{c}_k \tilde{z}_k \qquad (27)$$

If the norm of this vector exceeds the given error tolerance we can continue the process by explicit computing new matrix-vector products (20) with

$$y_k = P^{-1} r_{k-1} \qquad k > K \qquad (28)$$

where P is a preconditioning matrix.

The generation of new vectors continues until the desired accuracy is achieved.

The described method to solve system (19) with known matrix-vector products (20) can be applied to the solving of systems (15). Matrix-vector products for new parameter s are computed using (17). The efficiency of the method in this case is achieved due to reducing of the total number of matrix-vector multiplications.

If there are no accumulated vectors ($K = 0$) this approach coincides with the original GCR method. Therefore the algorithm based on the described approach is referred further as Multifrequency GCR algorithm.

In comparison with original GMRES method the original GCR method has the following shortcomings [13]:

1) in addition to orthogonalization (23) it requires extra operations to compute vectors $\tilde{y}_k$ (24);

2) the breakdown can occur during orthogonalization when current vector z_k is linearly dependent on previous basis vectors.

To reduce computational efforts we propose to save in memory scalar products used in (23), (24) as entries of upper triangular matrix H

$$h_{ij} = (\tilde{z}_i, z_j) \qquad i \le j \qquad (29)$$

This matrix defines linear transform of orthonormalized basis vectors $\tilde{z}_k$ to initial vectors z_k and similar relation between vectors y_k and $\tilde{y}_k$:

$$Z = \tilde{Z} H^T \qquad Y = \tilde{Y} H^T \qquad (30)$$

So we can exclude computations (24) and obtain coefficients c_k for expression (21) by the solving of linear system

$$Hc = \tilde{c} \qquad (31)$$

where $c = [c_1, \ldots, c_K]^T, \tilde{c} = [\tilde{c}_1, \ldots, \tilde{c}_K]^T$.

To overcome the breakdown problem the following operations are implemented:

- if linear dependence in orthogonalization occurs for vector z_k saved in memory then this vector must be skipped;

- if it occurs for vector obtained by explicit matrix-vector multiplication then orthogonalization step must be repeated (if needed more than once) by computing next Krylov subspace vector

$$z_k = (AP^{-1})^n z_k \qquad (32)$$

where n is the first integer number for which breakdown does not occur.

Iterations (32) can be performed by replacing (28) with

$$y_k = P^{-1} A y_k = P^{-1} z_k \qquad (33)$$

It can be shown that this modification avoids breakdown in the case of exact arithmetic.

Taking into account these proposed modifications the resulting Multifrequency Minimal Residual (MMR) algorithm for solving (15) can be presented in the following pseudocode form.

Multifrequency Minimal Residual (MMR) Algorithm

```
n = 0 // Number of saved matrix-vector products
For m = 1,...,M // Repeat for frequency index
  k = 1// Index of basis vector
  i_k = 1 // Memory index of current basis vector
  breakdown = NO // breakdown flag
  r = b_m // Residual vector
  Do
    If i_k > n // Perform and save matrix-vector product
      If breakdown = NO y_k = P^-1 r_k-1
      else y_k = P^-1 w
      n = n + 1
      z'_n = A'y_n      z''_n = A''y_n
    z_k = z'_i_k + s_m z''_i_k // Matrix-vector product
    w = z_k // Save vector to use in breakdown case
    For j = 1,...,k - 1 // Orthogonalize basis vector
      h_jk = (z_j, z_k)
      z_k = z_k - h_jk z_j
    If ||z_k|| < ε // breakdown
      breakdown = YES
      i_k = i_k + 1 // Increase current memory index
    else
      breakdown = NO
      z_k = z_k/||z_k|| //Normalize basis vector
      c_k = z_k r_k // Project residual on basis vector
      r = r - c_k z_k // Update r
      k = k + 1 // Increase basis vector index
      i_k = i_k-1 + 1 // Set current memory index
  While ||r_k|| > tolerance

  d = H^-1 c // Solve upper triangle system
```

$$x^{(m)} = \sum_{j=1}^{k-1} d_j y_{i_j} \quad \text{// Compute solution}$$

end

In comparison with Recycling GCR algorithm [4] presented MMR algorithm has the following advantages:

1) MMR algorithm does not impose any restrictions on matrices A', A'' and allows to exploit arbitrary frequency-dependent preconditioner,

2) MMR algorithm requires less number of operations due to elimination of additional linear transforms (24),

3) MMR algorithm avoids breakdowns.

Due to using the efficient algorithm of matrix-vector multiplication in time domain [7] the computational efforts for obtaining two vectors needed in the MMR algorithm are practically equal to the cost of one matrix-vector multiplication.

Note that proposed approach can be easily extended to circuits with distributed models. For such circuits the form of HB matrix is

$$A(s) = A' + sA'' + Y(s) \tag{34}$$

where $Y(s)$ is a harmonic admittance matrix of distributed models. In this case to apply MMR algorithm the computations by (17) are replaced by

$$z(s) = A(s)y = z' + sz'' + Y(s)y \tag{35}$$

The matrix Y contributes to diagonal blocks of matrix A only and as a rule is very sparse. Due to sparseness of this matrix the additional computational efforts are sufficiently small

4. Experimental Results

To estimate the numerical efficiency of the proposed Multifrequency Minimal Residual (MMR) algorithm in comparison with the original GMRES algorithm the periodic small signal analysis has been performed for three circuit examples (Table 1).

The frequency responses are presented in figures 1,2 for two first circuit examples. The obtained output frequency components of $|\omega + k\Omega|$ (k=-4,-3,-2,-1,0) versus input small signal frequency ω are given in fig.1 for one transistor bjt mixer. Here $\Omega = 1MHz$. The similar frequency characteristics for frequency converter with $\Omega = 140MHz$ are shown in fig. 2.

Table 1 contains time spents for calculation of characteristics using GMRES and MMR algorithms. The column 2 contains the number of harmonics, column 3 shows corresponding system orders. The column 4 contains CPU time for standard GMRES technique, column 5 shows speedup factors of MMR algorithm with respect to the standard GMRES technique. Column 6 gives the ratio of number of matrix-vector multiplication of GMRES and MMR algorithms. As we can see the reduction factor of number of matrix-vector operations (column 6) grows with increasing of size of the circuit. This provides more speedup for larger circuits (column 5).

The fourth example is the circuit consisting of Gilbert mixer, followed by filter and amplifier. This circuit contains 17 transistors, 47 resistors, 30 capacitors and 5 inductors. Here $\Omega = 1GHz$. Table 2 shows the computational efforts for various number of frequency points. The dependence is graphically presented in the Fig. 3. Note that the efficiency of MMR algorithm grows with increasing of number of frequency points.

These results confirm the computational efficiency of the proposed approach to frequency sweeping analysis.

Table 1 Computational efforts

V_{LO} (V)	K_h	system order	t_{gmres} (sec)	t_{gmres} / t_{mmr}	Nmv_{gmres}/ Nmv_{mmr}
Simple one transistor bjt mixer [16] (11 circuit variables)					
0.03	20	451	21	2.5	10
Frequency converter [5] (16 circuit variables)					
0.5	40	1296	64	3.0	27
Gilbert mixer (59 circuit variables, 6 transistors, 29 resistors, 28 capacitors, 3 inductors)					
0.4	20	2419	120	4.1	25
	40	4779	220	3.7	25
	60	7139	340	3.55	25

Table 2 Computational efforts for circuit 4 (121 variables, K_h=20)

number of frequency points	Nmv_{gmres}/ Nmv_{mmr}	t_{gmres} (sec)	t_{gmres}/ t_{mmr}
10	10	110	3.1
30	25	280	3.8
50	40	460	4.2

5. Conclusion

This paper presents a new efficient numerical Krylov subspace technique to support frequency sweeping for small signal periodic analysis based on harmonic balance. It was shown that this technique is suitable to sequential solving of linear problems where the matrix is a function of frequency. The simulation of typical RF circuits show that the new algorithm significantly increases the efficiency of periodic small-signal HB analysis.

6. References

[1] Cadence. RF Spectre User's Guide, 1998.

[2] Agilent. User's Guide. Advanced Design System 1.5, December 2000.

[3] R. Telichevesky, K. Kundert, "Receiver characterization using Periodic Small-Signal Analysis", in *Proc. of the IEEE Custom Integrated Circuits Conf.*, May 1996, pp. 449-452.

[4] R. Telichevesky, K. Kundert, J. White,"Efficient AC and Noise Analysis of Two-Tone RF Circuits", in *Proc. IEEE Design Automation Conference*, pp. 292-297, 1996.

[5] M. Okumura, T. Sugawara, and H. Tanimoto, "An efficient small signal frequency analysis method for nonlinear circuits with two frequency excitations." *IEEE Trans. of Computer-Aided Design of Integrated Circuits and Systems*, vol. 9, no. 3, pp. 225-235, March 1990.

[6] M. Okumura, H. Tanimoto, T. Itakura, T. Sugawara, "Numerical Noise Analysis for Nonlinear Circuits with a Periodic Large Signal Excitation Including Cyclostationary Noise Sources", *IEEE Trans. on Circuits and Systems - I*, vol. 40, N 9, pp. 581-590, 1993.

[7] P. Feldman, B. Melville, D. Long, "Efficient frequency domain analysis of large nonlinear analog integrated circuits", in *Proc. of the IEEE Custom Integrated Circuits Conf.*, May 1996, p. 461-464.

[8] B. Melville, P. Feldman, J. Roychowdhury, "Efficient multi-tone distortion analysis of analog integrated circuits", in *Proc. of the IEEE Custom Integrated Circuits Conf.*, May 1995, pp. 241-244.

[9] P. Heikilla, M. Valtonen, and T. Veijola, "Harmonic balance of nonlinear circuits with multitone excitation", *Proc. 10th European Conference on Circuit Theory and Design*, Vol. II, pp. 802-811, Copenhagen, 1991.

[10] R. Telichevesky, K. S. Kundert, J. K. White. "Efficient Steady-State Analysis based on Matrix-Free Krylov-Subspace Methods." In *Proc. 32 IEEE Design Automation Conference*, 1995, pp. 480-484.

[11] B. Yang, D. Feng "Efficient finite-difference method for quasi-periodic steady-state and small signal analyses," in *Proc. of International Conf. on Computer-Aided Design*, pp. 272-276, 2000.

[12] D. Feng, J. Phillips, K. Nabors, K. Kundert, J. White, "Efficient Computation of quasi-periodic circuit operating conditions via a mixed frequency/time approach", in *Proc. of 36 IEEE Design Automation Conference*, pp. 635-640, June 1999.

[13] Y. Saad, *Iterative methods for Sparse Linear Systems*, PWS publishing company, 1996.

[14] K.S. Kundert, J. White, A. Sangiovanni-Vincentelli, *Steady-State Methods for Simulating Analog and Microwave Circuits*, Kluwer Academic Publishers, Boston, 1990.

[15] K.S. Kundert, "Introduction to RF Simulation and Its Application", *IEEE J. of Solid-State Circuits*, v.34, no 9, 1999, pp. 1298-1319.

[16] A. Vladimirescu, *The Spice book*, John Wiley & Sons Inc., New York, 1994.

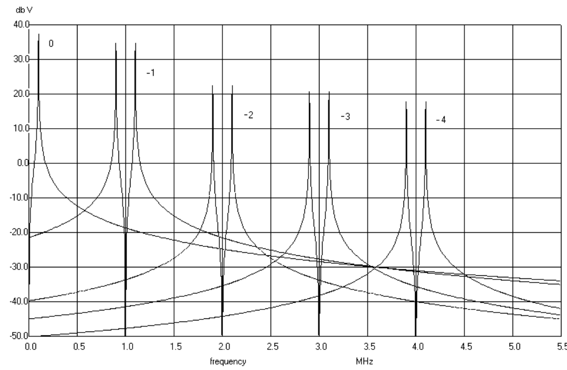

Figure 1. Output frequency components of $|\omega + k\Omega|$ versus input frequency ω for one transistor mixer circuit

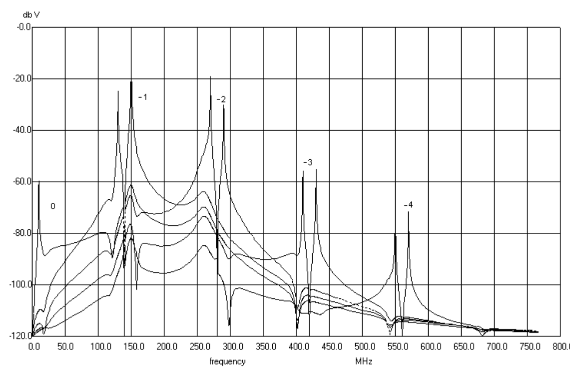

Figure 2. Output frequency components of $|\omega + k\Omega|$ versus input frequency ω for frequency converter

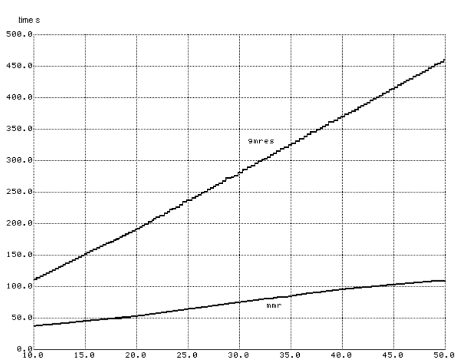

Figure 3. Computational efforts versus number of frequency points