Projective Convolution: RLC Model-Order Reduction Using the Impulse Response

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Abstract

Projective convolution (PC) is a provably passive and numerically well-conditioned model-order reduction technique for large RLC circuits including those with floating capacitors or inductor loops. Unlike moment-matching which operates in the frequency domain, PC is positioned squarely in the time domain: it matches the impulse response of a circuit by projecting with the Krylov space formed by solving the discretized differential equations of the circuit. PC gives excellent results for coupled lines, large RLC meshes, and clock trees.

Introduction

Most model-order reduction techniques work in the framework of matching moments in the complex frequency domain--an idea introduced by Pileggi and Rohrer in their 1990 seminal paper on AWE [1]. In 1992, Chiprout and Nakhla extended the idea by doing moment expansions at several points in the complex plane [2]. In 1994, Feldmann and Freund introduced PVL, which uses a Lanczos procedure to obtain a well-conditioned basis for the moments [3]. In 1995, the same authors proposed the idea of block reduction [4], while Silveira *et al.* introduced Arnoldi as an alternative to the Lanczos process [5], and Kerns *et al.* broached the key concept of congruence transform to preserve passivity[6]. Recent papers have extended congruence techniques to multi-point expansions and stable reduction of circuits with inductance [7] [8] [9] [10].

Projective convolution, by contrast, attempts to match a circuit's impulse response directly rather than matching moments in the frequency domain. This is an enviable goal, since applications like timing verification are primarily interested in a time-domain response.

We will show how the Krylov spaces that arise naturally in solving recurrence relations can be used advantageously to form a projection matrix. Applying this idea to the discretization of a circuit's differential equations, we get **projective convolution--**projective because it forms a reduced system by projection (to be defined shortly); convolution because, like convolution, it computes the response to an arbitrary stimulus from a circuit's impulse response. As we shall see, PC projects the circuit equations with a matrix formed by sampling the impulse response of the circuit.

Circuit Equations

The equations for an N+1 terminal, time-invariant, linear circuit can be written

$$(C\frac{d}{dt}+G)x = Bj,\;\; u = B^T x \qquad (1.1)$$

where $j\in L^N$ and $u\in L^N$ are port current and voltage vectors, $x\in L^m$ is a vector of nodal voltages and selective branch currents, $B\in R^{mxN}$ is the port incidence matrix, and $C\in R^{mxm}$ and $G\in R^{mxm}$ are the susceptance and conductance matrices. Here, L is the space of real-valued functions of time defined for t>=0. If C and G are symmetric matrices, system W is called **self-adjoint**; circuits consisting only of capacitors and resistors are self-adjoint. If C and G are positive semidefinite, then W is called **positive semidefinite**; the modified nodal analysis equations of RLC circuits are positive semidefinite.[1]

For brevity, we shall denote system (1.1) as $W^m(C,G,B)$. The dimension m of the state vector x is called the **order** of the system. Our goal is to replace a system $W^m(C,G,B)$ by another system $\tilde{W}^{\tilde{m}}(\tilde{C},\tilde{G},\tilde{B})$ of lower order, i.e. $\tilde{m} < m$, such that important characteristics of the two systems are similar. We call this the **reduction** problem.

System Properties

We unfold our subject in stages, beginning with the concepts of reciprocity, stability, and passivity, since these are properties we want to preserve during reduction.

Definition 2.1. The **transfer function** of a system $W^m(C,G,B)$ is the NxN rational matrix

$$Z(s) = B^T(Cs+G)^{-1}B \qquad (2.1)$$

where s is the complex frequency.

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[1] The scalar $a = x^T Ax$ depends only on the symmetric part of *A*, for $a = \frac{1}{2}(a+a^T) = \frac{1}{2}x^T(A+A^T)x$. *A* is positive semidefinite if $a \geq 0$ for all vectors *x*.

Definition 2.2. A system *W* is **reciprocal** if its transfer function Z(s) is symmetric, i.e. $Z^T(s)=Z(s)$.

Definition 2.3. A system *W* is **stable** if all poles of its transfer function Z(s) are in the *open* left-half plane. It is **asymptotically stable** if the poles are in the *closed* left-half plane.

Definition 2.4. A system *W* is **passive** if the rational transfer function Z(s) satisfies (see [11]):

(1) $Z(s^*) = [Z^T(s)]^*$ for all complex s, where * means conjugate transpose, and

(2) $z^*(Z(s)+Z^*(s))z \geq 0$ for all complex vectors z and complex s with Re(s)>0.

A passive system is asymptotically stable since it cannot have poles in the open right-half plane; moreover, interconnections of passive systems are passive.

Theorem 2.1. A self-adjoint system *W* is reciprocal.

Proof. $\{B^T(Cs+G)^{-1}B\}^T = B^T(C^Ts+G^T)^{-1}B$.

Theorem 2.2. A positive-semi-definite system *W* is passive. Proof. We show that the two conditions of definition 2.4 are satisfied. Let $W^m(C,G,B)$ be the system, Z(s) its transfer function. Since C, G, and B are real, conjugating Z(s) merely conjugates s, so (1) is trivially true. Letting T=Cs+G, condition (2) requires $z^*B^T(T^{-1}+T^{-*})Bz \geq 0$ where $T^{-*} \equiv [T^{-1}]^*$. But this is the same as $z^*B^TT^{-*}(T^*+T)T^{-1}Bz$ or

$$w^*(s(C+C^T)+jw(C-C^T)+(G+G^T))w \qquad (2.2)$$

where $w=T^{-1}Bz$ and $s = s + jw$. Since $g = w^*Cw$ is a scalar, $w^*(C-C^T)w = g - g^T = 0$. Hence (2.2) is greater or equal to zero because C and G are positive semi-definite by assumption.

Projection Methods

We now show that orthogonal projection preserves reciprocity and passivity.

Definition 3.1. A **orthogonal projection** of a system $W^m(C,G,B)$ by **projection method** P_V, where $V\in R^{mxq}$, is a system $W^q(\tilde{C},\tilde{G},\tilde{B})$ with matrices

$$\tilde{C}=V^TCV \quad \tilde{G}=V^TGV \quad \tilde{B}=V^TB \qquad (3.1)$$

q is the **dimension** of P_V, which is **proper** if q<m and rank(V)=q.

The reader will recognize orthogonal projection to be none other than the classic **Galerkin process.** In (1.1) we approximate the state vector x by a vector in the column space of V, i.e. we let $x \approx Vy$ for some $y\in L^q$. In general, equation (1.1) will no longer be exactly satisfied; there will be some residue $r\in L^m$ defined by $r \equiv (C\frac{d}{dt}+G)Vy - Bj$. In the Galerkin method, we impose the condition that r be orthogonal to *col-span*(V) at all times. In other words, $V^Tr=0$. This gives us system $W^q(\tilde{C},\tilde{G},\tilde{B})$ defined by (3.1).

Definition 3.2. Two orthogonal projectors are **equivalent** if the columns of their projection matrices span the same space.

Projecting a system *W* with equivalent projectors results in essentially the same reduced system--i.e. the projected systems have the same impulse responses *in exact arithmetic*. But in practice, V^TV should not be too ill-conditioned; ideally, $V^TV\approx I$.

Lemma 3.1. The orthogonal projection of a self-adjoint system is self-adjoint.

Proof. $\tilde{C}^T=(V^TCV)^T=V^TC^TV=V^TCV=\tilde{C}$, since $C^T=C$. $\tilde{G}^T=\tilde{G}$ similarly.

Lemma 3.2. The orthogonal projection of a positive-(semi)definite system is positive-(semi)definite.

Proof. For all $x\in R^q$, $x^T\tilde{C}x = x^TV^TCVx = w^TCw \geq 0$, where $w=Vx$, since C is positive semi-definite. The same argument applies to G.

An immediate consequence of these lemmas and theorems 2.1 and 2.2 of the last section is:

Theorem 3.1. The orthogonal projection of a self-adjoint system is reciprocal; the orthogonal projection of a positive semi-definite system is passive.

Krylov Spaces

Definition 4.1. The **Krylov space** generated by matrices $A\in R^{mxm}$ and $B\in R^{mxN}$ is $K_n(A,B)$=*col-span*{[B,AB,...,A^{n-1}B]}. Here [B,AB,...,A^{n-1}B] is the matrix whose columns are the columns of B followed by the columns of AB, etc.

Definition 4.2. Any matrix V such that *col-span*(V)=$K_n(A,B)$ is called a **Krylov projector** .

Krylov spaces arise naturally from solving recurrence relations.

Theorem 4.1. The two-term matrix recurrence relation

$$EX_0 = B,\ EX_n + GX_{n-1} = 0 \qquad (4.1)$$

where $E,G\in R^{mxm}$ and $B\in R^{nxN}$ has solution sequence $\{X_n=(-E^{-1}G)^nE^{-1}B,\ X_n\in R^{mxN}\}$. Hence *col-span*{ [$X_0,X_1,...,X_n$]}=$K_n(-E^{-1}G,E^{-1}B)$; in other words, the matrix [$X_0,X_1,...,X_n$] compounded from the columns of the solution sequence of (4.1) is a Krylov projector.

Two-term recurrence relations have the desirable property that the solution sequence can be **orthogonalized on the fly**: at each step we can replace X_{n-1} by a matrix Y_{n-1} such that *col-span*(Y_{n-1})=*col-span*(X_{n-1}), then solve $EX_n + GY_{n-1} = 0$. This procedure preserves column spaces,

$$colspan\{[Y_0, \ldots, Y_n]\} = colspan\{[X_0, \ldots, X_n]\}$$

Let us denote a two-term recurrence relation like (4.1) by R(E,G,B). Recurrence relations can be projected just like system equations: the orthogonal projection of R by projector V is the new recurrence relation $\tilde{R}(V^T EV, V^T GV, V^T B)$. We shall use the next result to prove theorem 5.1.

Theorem 4.2. Let R(E,G,B) be (4.1) with solution sequence $\{X_i\}$ and let $P_V R = \tilde{R}(V^T EV, V^T GV, V^T B)$ be its projection with solution sequence $\{\tilde{X}_i\}$. If the projection matrix V spans the initial n+1 terms in the solution sequence of R, that is, if *col-span*(V)⊃*col-span*{[$X_0, X_1, \ldots, X_n$]}, then

$$X_i = V\tilde{X}_i \quad i = 0, \ldots, n \tag{4.2}$$

Proof. Because of the *col-span* assumptions for V, we know that there exist matrices $\tilde{X}_i$ satisfying (4.2). These $\tilde{X}_i$ satisfy $\tilde{R}$ since X_i satisfy R, e.g. $V^T EV\tilde{X}_i + V^T GV\tilde{X}_{i-1} = V^T(EX_i + GX_{i-1}) = V^T(0) = 0$. There is no other possibility, since the solution sequences of R and $\tilde{R}$, if they exists (E and $V^T EV$ must be nonsingular), are unique.

Time Domain Response Matching

We have seen that if we reduce an N-port, self-adjoint, positive-semidefinite circuit by orthogonal projection, the projected system is both reciprocal and passive. One way to choose the projection matrix is to orthogonalize on the fly solutions to a two-term recurrence relation. But what recurrence relation should we use? To find the time domain response of system (1.1), we must solve a system of differential equations. But this is commonly done by replacing (1.1) with a discretized system of recurrence relations! Why not use these recurrence relations to get our projection matrix?

Definition 5.1. When $W^m(C,G,B)$ is replaced by the recurrence system

$$R\left(\frac{C}{h} + aG, \frac{-C}{h} + (1-a)G, B\right) \tag{5.1}$$

where α is a constant between 0 and 1, the replacement is called a **discretization**. If the solution sequence for (5.1) is $\{X_j\}$, then the sequence

$$\{U_j = B^T(aX_j + (1-a)X_{j-1}),\ X_{-1} \equiv 0\} \tag{5.2}$$

is the **numerical impulse response** of *W*. The integration method is **Forward Euler** if α=0, **Trapezoidal** if α=1/2, and **Backward Euler** if α=1. Denote the mapping from (1.1) to (5.1) by D_a^h, the discretization operator. Likewise, denote the numerical impulse response operator by *NIR*, so that $\{U_j\} = NIR(W, D_a^h)$.

Here is the main result of projective convolution theory:

Theorem 5.1. Given a system *W*, let $\{X_j\}$ be the solution sequence of $D_a^h W$. Let V be a full column rank matrix such that *col-span*([$X_0, \ldots, X_n$]) Ì *col-span*(V). Then the initial n+1 terms of the numerical impulse responses of *W* and $P_V W$ match.

Proof. Let $\{X_j\}$ be the solution sequence of $D_a^h W$, and $\{\tilde{X}_j\}$ the corresponding solution sequence of $P_V D_a^h W$. By Theorem 4.2 the initial n+1 terms of $D_a^h W$ and its projection stand in the relation $X_j = V\tilde{X}_j \quad j = 0, \ldots, n$. But then for these terms

$$NIR(W, D_a^h)_j = B^T(aX_j + (1-a)X_{j-1}) =$$
$$B^T V(a\tilde{X}_j + (1-a)\tilde{X}_{j-1}) = NIR(P_V W, D_a^h)_j$$

where $NIR(W, D_a^h)_j$ is the j'th term in the numerical impulse response of *W* using discretization $D_a^h W$.

Definition 5.2. Orthogonal projection by a matrix V satisfying the conditions of theorem 5.1 is called **projective convolution**.

Loosely speaking, theorem 5.1 says that projective convolution preserves the impulse response. This is true strictly only for the discretization method and step-size used in forming V, but in practice PC is surprisingly effective. Perhaps this is because PC is a Krylov method: Krylov spaces have an uncanny ability to approximate matrix functions (like the matrix function $e^{At}B$ that is the impulse response of $\dot{x} = Ax + Bu$).

There is the practical issue of how many terms of the solution sequence $\{X_j\}$ to use in forming V. We get adaptive error control if we add terms until two successive reduced systems agree in their response to within a given tolerance. It is important to realize that the discretization step size h used to generate V can be quite large in practice (see the examples).

Relation To Moment Matching

We next uncover a relation between projective convolution and the traditional method of moments. Recall that moment matching substitutes $X(s) = M_0 + M_1\tilde{s} + M_2\tilde{s}^2 + \ldots$, where $\tilde{s} = s - s_0$, into the Laplace transform of (1.1):

$$\{C\tilde{s} + (Cs_0 + G)\}(M_0 + M_1\tilde{s} + ...) = B \qquad (6.1)$$

Equating powers of $\tilde{s}$ yields recurrence relation $R_M(Cs_0+G,C,B)$ whose solution sequence $\{M_j\}$ are called **moments** about s_0. It then solves R_M for the first n+1 moments, builds up a well-conditioned matrix V_M such that *col-span*(V_M)=*col-span*$\{[M_0,...,M_n]\}$, and orthogonally projects. This in our nomenclature is the modern method of moments [10].

Theorem 6.1. Projective convolution using backward Euler discretization is equivalent to reduction by moment matching about the frequency s_0=1/h.

Proof. Moment matching leads to the recurrence system $R_M(Cs_0+G,C,B)$; with 1/h=s_0, D_a^h leads to $R_{BE}(Cs_0+G,-Cs_0,B)$. From (4.1), the solution sequences $\{X_j\}$ of R_{BE} and $\{M_j\}$ of R_M are related by $X_j = (-s_0)^{j-1} M_j$; being proportional, *col-span*$([X_0,...,X_n])$=*col-span*$([M_0,...,M_n])$ and the orthogonal projections of the two methods are equivalent.[2]

Examples

We draw our examples from the realm of printed circuit board networks rather than VLSI circuits because we feel these wider-bandwidth, inductive circuits better showcase PC's stability and numerical conditioning as well as its easy accommodation of inductance. Our experience is that PC is also highly effective with VLSI circuits, but convergence is much quicker (i.e. fewer *NIR* terms are required for a given accuracy).

Transmission Line Mesh

First consider an 8x12 mesh of 1 inch, 50 ohm transmission lines modeled by 3 RLC sections per inch; this leads to an system of 1177 variables. Figure 1 shows a 2x3 version of the mesh; the actual circuit has 96, rather than 6, cells.

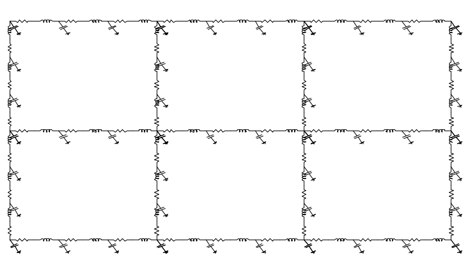

Figure 1. A 2 x 3 mesh

[2] Interestingly, before the introduction of Arnoldi and Lanczos methods, one proposal for improving the stability of AWE was to scale the frequency s by some $s_0 \approx 1/h$, h being the dominant time constant of the circuit. Projective convolution automatically does this scaling.

A 50 ohm, 0.5ns saturating-ramp driver is attached to one corner of the mesh, while the response at the center and opposite corner are monitored. Fixing h=1.0 ns, we reduce the circuit to a 21 variable system using α=0.5 (Trapezoidal) and α=1 (backward Euler). See Figure 2. While the $D_{1/2}$ response is almost indistinguishable from the unreduced system, D_1 exhibits pre-arrival-time ripple at the far corner.

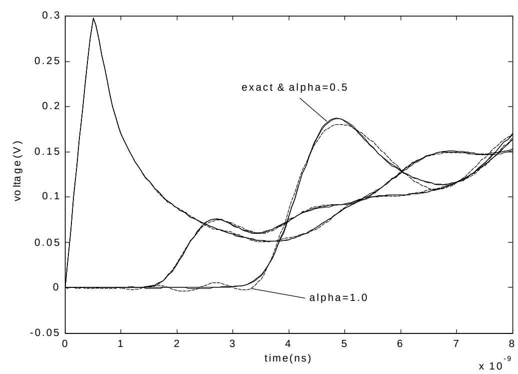

Figure 2. Response for 8 x 12 mesh

Balanced Clock Tree

Next, consider a 5-level balanced tree such as might be encountered in routing a clock; each branch is a 2 inch, 50 ohm line. With a 1 pF load at each leaf, the root is driven by a 50 ohm, 0.5 ns ramp source.

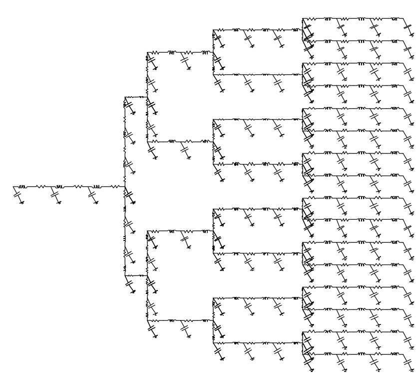

Figure 3. 5-level balanced tree

Figure 4 compares a $D_{1/2}$ (h=1.0 ns, order 13) projected system to the unreduced system of 621 variables; the responses, given for each level of the tree, are almost indistinguishable.

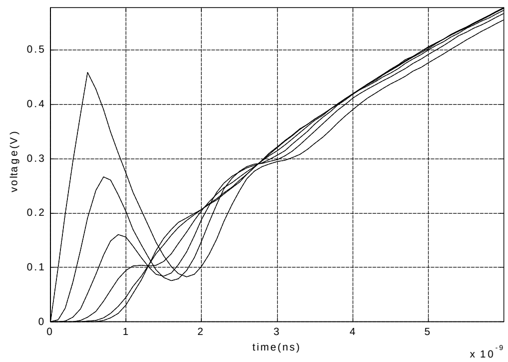

Figure 4. Response of clock tree

Coupled Lines

The last example is a pair of 4 inch lines modeled by 40 coupled RLC sections (R=0.014 ohm, L=1 nH, C=0.4 pF, and C_m=0.08 pF per section). Figure 5 plots the response at the four ports of the original (order 162) and a $D_{1/2}$ projected (h=1 ns, order 16) system; again, the responses nearly coincide.

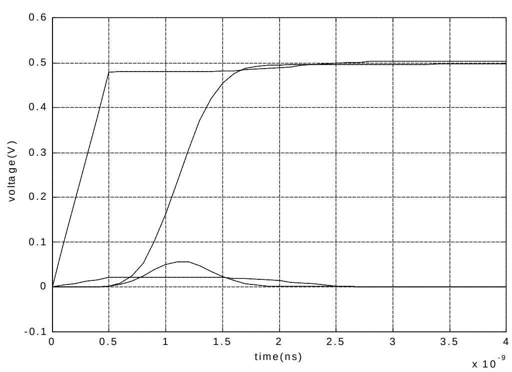

Figure 5. Two coupled lines

Conclusion

Conceptually and practically projective convolution has much to recommend it. Practically, it replicates almost perfectly the responses of coupled lines, meshes, trees--even when low-loss transmission lines are used, which is certainly the harder case. By projecting into a Krylov subspace derived from the impulse response of a circuit, it is, by construction, made to reproduce that response.

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