# Implicit index-aware model order reduction for RLC/RC networks


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***Abstract*—This paper introduces the implicit-IMOR method for differential algebraic equations. This method is a modification of the Index-aware model order reduction (IMOR) method proposed in our earlier papers which is the explicit-IMOR method. It also involves first splitting the differential-algebraic equations (DAEs) into differential and algebraic parts using a basis of projectors. In contrast with the explicit-IMOR method, the implicit-IMOR method leads to implicit differential and algebraic parts. We demonstrate the implicit-IMOR method using the RLC/RC networks, but it can also be applied to other problems which lead to differential-algebraic equations.**


## I. Introduction

Consider a linear RLC electric network, that is, a network which connects linear capacitors, inductors and resistors, and independent current sources $\boldsymbol{\imath}(t) \in \mathbb{R}^{n_I}$. The unknowns which describe the network are the node potentials $\boldsymbol{e}(t) \in \mathbb{R}^n$, and the currents through inductors $\boldsymbol{\jmath}_L(t) \in \mathbb{R}^{n_L}$. Following the formalism of Modified Nodal Analysis (MNA) [1], we introduce: the incidence matrices $\mathbf{A}_C \in \mathbb{R}^{n,n_C}$, $\mathbf{A}_L \in \mathbb{R}^{n,n_L}$ and $\mathbf{A}_R \in \mathbb{R}^{n,n_G}$, which describe the branch-node relationships for capacitors, inductors and resistors; the incidence matrix $\mathbf{A}_I \in \mathbb{R}^{n,n_I}$, which describe this relationship for current sources. Then this leads to a descriptor system for the unknown $\boldsymbol{x} = (\boldsymbol{e}, \boldsymbol{\jmath}_L)^\top$ given by

$$\underbrace{\begin{pmatrix} \mathbf{A}_C \boldsymbol{\mathcal{C}} \mathbf{A}_C^\top & 0 \\ 0 & \boldsymbol{L} \end{pmatrix}}_{\mathbf{E}} \frac{d\boldsymbol{x}}{dt} = \underbrace{\begin{pmatrix} -\mathbf{A}_R \boldsymbol{\mathcal{G}} \mathbf{A}_R^\top & -\mathbf{A}_L \\ \mathbf{A}_L^\top & 0 \end{pmatrix}}_{\mathbf{A}} \boldsymbol{x} + \underbrace{\begin{pmatrix} -\mathbf{A}_I \\ 0 \end{pmatrix}}_{\mathbf{B}} \boldsymbol{\imath}, \tag{1}$$

with consistent initial data

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{2}$$

Here, $\boldsymbol{\mathcal{C}} \in \mathbb{R}^{n_C,n_C}$, $\boldsymbol{L} \in \mathbb{R}^{n_L,n_L}$ and $\boldsymbol{\mathcal{G}} \in \mathbb{R}^{n_G,n_G}$ are the capacitance, inductance and conductance matrices, which are assumed to be symmetric and positive-definite. Note that we consider a network of only current sources for simplicity but also voltages sources can be used. If $\mathbf{E}$ is singular, (1) is a differential algebraic equation (DAE) otherwise it is an ordinary differential equation (ODE). In this paper we assume that $\mathbf{E}$ is singular, thus we are considering DAEs. The dimension of the DAE system (1) is $N = n + n_L$.


**978-3-9815370-2-4/DATE14/©2014 EDAA**


System (1) is the state equation which describes the system's dynamics. The output equation which describes the observation is given by

$$\boldsymbol{y} = \mathbf{C}^T \boldsymbol{x}, \tag{3}$$

where $\mathbf{C} \in \mathbb{R}^{N,\ell}$. Combining (1) and (3), we obtain a control problem given by

$$\mathbf{E}\boldsymbol{x}' = \mathbf{A}\boldsymbol{x} + \mathbf{B}u \tag{4a}$$

$$\boldsymbol{y} = \mathbf{C}^T \boldsymbol{x}, \tag{4b}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{N,N}$, $\mathbf{B}^{N,m}$. If $m, \ell > 1$, the system (4) is called multiple-input multiple-output (MIMO), and if $m = \ell = 1$ it is called single-input single-output (SISO). Taking the Laplace transform of system (4), we obtain

$$\boldsymbol{Y}(s) = \mathbf{H}(s)\boldsymbol{U}(s) + \mathbf{G}(s)\boldsymbol{x}_0. \tag{5}$$

The matrix function $\mathbf{H}(s) := \mathbf{C}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}$, is traditionally called the transfer function if we assume vanishing initial condition $\boldsymbol{x}_0 = 0$ while the function $\mathbf{G}(s) := \mathbf{C}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{E}$. In this paper, we consider index-1 systems, thus we shall always obtain the transfer function $\mathbf{H}(s)$.

In practice $N \gg m, \ell$, thus solving (4) in real time can be computationally expensive. This is an attractive feature for Model Order Reduction (MOR) [6]. The goal of model order reduction is to replace the original dynamics in (4) by a model of the same form but with a much smaller state space dimension. Thus, we seek a reduced-order model

$$\mathbf{E}_r \boldsymbol{x}_r' = \mathbf{A}_r \boldsymbol{x}_r + \mathbf{B}_r \boldsymbol{u}, \tag{6a}$$

$$\boldsymbol{y}_r = \mathbf{C}_r^T \boldsymbol{x}_r, \tag{6b}$$

with matrices $\mathbf{E}_r, \mathbf{A}_r \in \mathbb{R}^{r,r}$, $\mathbf{B}_r \in \mathbb{R}^{r,m}$, and $\mathbf{C}_r \in \mathbb{R}^{r,\ell}$ such that $r \ll N$ and the output approximation error $\boldsymbol{y} - \boldsymbol{y}_r$ is small with respect to a specific norm over a wide range of inputs $\boldsymbol{u}$. In the frequency domain, this means that the transfer function of (6) is given by $\mathbf{H}_r(s) := \mathbf{C}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$ approximates $\mathbf{H}(s)$ well, i.e., the error $\mathbf{H}(s) - \mathbf{H}_r(s)$ is small in a certain system norm. The reduction methods can be classified in into two: spectral and Krylov-subspace based methods. These methods are extensively discussed in [6], [10]. In this paper we shall focus on the Krylov-subspace based methods which exploit the use of Krylov subspace iterations to achieve system approximation by moment matching. Among these methods are PRIMA [7], the structure preserving version SPRIM [8] and so on. The most important advantages offered by PRIMA are: the applicability to MIMO systems and passivity preservation. However the two main limitations of PRIMA are that it does not preserve the MNA structure of the original system and the index of the system, i.e., It leads to ODE reduced-order models. Moreover, PRIMA leads to wrong reduced-order models for DAEs of higher index [2]. The problem of not preserving the MNA structure was solved by its structure preserving version SPRIM. However, index problem was not solved till now. In [3] and [2] we proposed a index-aware model order reduction (IMOR) method which preserves the index of the system and can be used to even reduce higher index DAEs. In this method we first split the DAE into differential and algebraic parts using projectors and their respective bases. Then we can use conventional MOR methods such as PRIMA, to reduce the differential part and then develop techniques to reduce the algebraic parts. However the explicit-IMOR method proposed in [3], [2] involves matrix inversion which may be computationally expensive. In this paper we modify the method by splitting the DAEs without matrix inversions, which we call the implicit-IMOR (IIMOR) method. This paper is organized as follows: Sect. II, we briefly discuss about the PRIMA method, then in Sect. III, we discuss about the decoupling of the RLC and RC networks without matrix inversions. Then in Sect. IV, we discuss the IIMOR method. Finally we carry out experiments using DAEs from electric power grid models and then the conclusions.

## II. Model Order Reduction

At the heart of model order reduction lies the desire to approximate the behavior of a large dynamical system in an efficient manner, so that the resulting approximation error is small [5]. Other requirements are: the preservation of important system properties, of its physical interpretation, and an efficient implementation. In other words, the reduced-order model must be computationally cheaper to solve than its original model. We replace the original system (4) for $\boldsymbol{x} \in \mathbb{R}^n$, with output $\boldsymbol{y} \in \mathbb{R}^\ell$, with the reduced system (6) for $\boldsymbol{x}_r \in \mathbb{R}^r$, with output $\boldsymbol{y}_r \in \mathbb{R}^\ell$. The unifying approach for obtaining a reduced-order model from an original system is via a Petrov-Galerkin projection: $\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$,
$\mathbf{B}_r = \mathbf{W}^T\mathbf{B}$ and $\mathbf{C}_r = \mathbf{V}^T\mathbf{C}$, where $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n,rm}$ are the matrices whose $r \ll N$ columns form bases for the relevant subspaces pertaining to the reduction method chosen. Model Order Reduction methods differ in the way the decomposition is performed, this in turn dictates how the projection matrices $\mathbf{V}$ and $\mathbf{W}$ are constructed. There are many MOR methods, but in this paper we shall focus on PRIMA [7] which is the most popular reduction method for electric circuits. In this method , one assumes the Galerkin projection, i.e $\mathbf{V} = \mathbf{W} \in \mathbb{R}^{n,rm}$. Then the projection $\mathbf{V}$ is constructed as follows: Choosing arbitrary expansion point $s_0 \in \mathbb{C}$, then we consider order-$r$ Krylov subspace generated by $\mathbf{M} = (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}$ and $\mathbf{R} = (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$ , that is $\mathcal{K}_r(\mathbf{M}, \mathbf{R}) = \operatorname{span}\{\mathbf{R}, \mathbf{M}\mathbf{R}, \cdots, \mathbf{M}^{r-1}\mathbf{R}\}, r \le n$ and denoted by $\mathbf{V} \in \mathbb{R}^{n,rm}$ the matrix of an orthonormal basis for $\mathcal{K}_r(\mathbf{M}, \mathbf{R})$ so that $\mathbf{V}^T\mathbf{V} = \mathrm{I}$. However the PRIMA method is not valid for DAEs of higher index and does not preserve the index of the DAE system. In [3] and [2], they proposed a new MOR method for index-1 and -2 DAEs respectively, which they called the IMOR method. This method involves first splitting the DAE into differential and algebraic parts. Then one can apply any model order reduction method on the differential part and also reduce the algebraic parts. In this paper, we call it the explicit-IMOR method. However the explicit- IMOR method involves matrix inversions which may be computationally expensive for very large systems, this motivated us to develop its no inversion version which we call the implicit-IMOR (IIMOR) method.

## III. Decoupling of RLC/RC networks

In this section, we introduce the implicit splitting of DAEs using projectors and their corresponding bases. This splitting is different from that proposed in [3], although the approach is almost the same. Here we consider the splitting of index-1 RLC/RC networks but the same procedure can be applied to any index-1 system. In order to decouple system (1) we need to first construct the matrix and projector chains of the matrix pencil $(\mathbf{E}, \mathbf{A})$ using the definition of tractability index as defined in [11]. Setting $\mathbf{E}_0 := \mathbf{E}, \quad \mathbf{A}_0 := \mathbf{A}$, further

$$\mathbf{E}_{j+1} = \mathbf{E}_j - \mathbf{A}_j\mathbf{Q}_j, \quad \mathbf{A}_{j+1} := \mathbf{A}_j\mathbf{P}_j, \quad j \ge 0, \qquad (7)$$

whereby $\mathbf{Q}_j$ denotes a projector onto the nullspace $\operatorname{Ker}\mathbf{E}_j$ and its complementary projector $\mathbf{P}_j = \mathrm{I} - \mathbf{Q}_j$. The sequence $\mathbf{E}_0, \mathbf{E}_1, \cdots$ , is known to become stationary, i.e., $\mathbf{E}_{\mu+j} = \mathbf{E}_\mu$, $j \ge 0$ , where $\mu$ is the tractability index, supposing the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is regular.

Assuming the matrix pencil $(\mathbf{E}, \mathbf{A})$ is regular, we can compute the tractability index of (4a). This can be done as follows:
We first set $\mathbf{E}_0 = \mathbf{E}, \quad \mathbf{A}_0 = \mathbf{A}$ :

$$\mathbf{E}_0 = \begin{pmatrix} \mathbf{A}_C\boldsymbol{\mathcal{C}}\mathbf{A}_C^\top & 0 \\ 0 & \boldsymbol{L} \end{pmatrix}, \quad \mathbf{A}_0 = \begin{pmatrix} -\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top & -\mathbf{A}_L \\ \mathbf{A}_L^\top & 0 \end{pmatrix}.$$

We then choose a projector $\mathbf{Q}_0$ such that $\operatorname{Im}\mathbf{Q}_0 = \operatorname{Ker}\mathbf{E}_0$ and its complementary projector given by $\mathbf{P}_0 = \mathrm{I} - \mathbf{Q}_0$. For this class of problem, we can just choose a projector $\mathbf{Q}_C$ that projects onto the kernel of $\mathbf{A}_C^\top$, and $\mathbf{P}_C = \mathbf{I} - \mathbf{Q}_C$. Then we can obtain:

$$\mathbf{Q}_0 = \begin{pmatrix} \mathbf{Q}_C & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{P}_0 = \begin{pmatrix} \mathbf{P}_C & 0 \\ 0 & \mathbf{I} \end{pmatrix}.$$

Then,

$$\mathbf{E}_1 = \mathbf{E}_0 - \mathbf{A}_0\mathbf{Q}_0 = \begin{pmatrix} \mathbf{A}_C\boldsymbol{\mathcal{C}}\mathbf{A}_C^\top + \mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\mathbf{Q}_C & 0 \\ -\mathbf{A}_L^\top\mathbf{Q}_C & \boldsymbol{L} \end{pmatrix},$$

$$\mathbf{A}_1 = \mathbf{A}_0\mathbf{P}_0 = \begin{pmatrix} \mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\mathbf{P}_C & -\mathbf{A}_L \\ \mathbf{A}_L^\top\mathbf{P}_C & 0 \end{pmatrix}.$$

It can easily be proved that if, we have the condition:

$$\ker(\mathbf{A}_C, \mathbf{A}_R)^\top = \{0\}, \qquad (8)$$

then we find that $\boldsymbol{x} \in \ker\mathbf{E}_1$ if and only if $\mathbf{Q}_C\boldsymbol{e} = 0$ and assuming $\boldsymbol{L}$ is a nonsingular matrix. Thus the condition (8) is equivalent to the index-1 condition,i.e., $\mathbf{E}_1$ is non-singular.

If the condition (8) is not satisfied, we need to iterate the procedure. In this case, we need to introduce a projector

$\mathbf{Q}_1$ onto the $\operatorname{Ker}\mathbf{E}_1$, satisfying the additional requirements $\mathbf{Q}_1\mathbf{Q}_0 = 0$. To do this, we need to first introduce a projector

$$\bar{\mathbf{Q}}_1 = \begin{pmatrix} \mathbf{Q}_{CR} & 0 \\ \boldsymbol{L}^{-1}\mathbf{A}_L^T\mathbf{Q}_C & 0 \end{pmatrix} \tag{9}$$

which also projects onto the $\operatorname{Ker}\mathbf{E}_1$, where $\mathbf{Q}_{CR}$ is a projector onto $\ker(\mathbf{A}_C, \mathbf{A}_R)^\top$. Then, the projector $\mathbf{Q}_1 = -\bar{\mathbf{Q}}_1(\mathbf{E}_1 - \mathbf{A}_1\bar{\mathbf{Q}}_1)^{-1}\mathbf{A}_1$ is a projector onto the $\operatorname{Ker}\mathbf{E}_1$, that satisfies the condition $\mathbf{Q}_1\mathbf{Q}_0 = 0$. Thus, we can find $\mathbf{E}_2 = \mathbf{E}_1 - \mathbf{A}_1\mathbf{Q}_1$ and $\mathbf{A}_2 = \mathbf{A}_1\mathbf{P}_1$. If $\mathbf{E}_2$ is nonsingular, then (4) is an index-2 system. In this paper, we assume that index-1 condition (8) is satisfied thus (4a) is an index-1 system. From this point, we assume $\mathbf{E}_1$ is nonsingular unless otherwise stated.

### A. *Index-1 RLC network*

In this section, we decouple index-1 systems of the form (4a). We first construct bases for projectors $\mathbf{Q}_C$ and $\mathbf{P}_C$ given by $\boldsymbol{q}_c \in \mathbb{R}^{n,n_1}$, $\boldsymbol{p}_c \in \mathbb{R}^{n,n_2}$ and their respective inverses are given by $\boldsymbol{q}_c^{*T} \in \mathbb{R}^{n_1,n}$, $\boldsymbol{p}_c^{*T} \in \mathbb{R}^{n_2,n}$, where $n = n_1 + n_2$. Thus the bases for projectors $\mathbf{Q}_0$ and $\mathbf{P}_0$ are given by $\boldsymbol{q} = \begin{pmatrix} \boldsymbol{q}_c \\ 0 \end{pmatrix} \in \mathbb{R}^{N,n_1}$, $\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_c & 0 \\ 0 & \mathbf{I} \end{pmatrix} \in \mathbb{R}^{N,n_2+n_L}$ and their respective inverses are given by $\boldsymbol{q}^{*T} = \begin{pmatrix} \boldsymbol{q}_c^{*T} & 0 \end{pmatrix} \in \mathbb{R}^{n_1,N}$, $\boldsymbol{p}^{*T} = \begin{pmatrix} \boldsymbol{p}_c^{*T} & 0 \\ 0 & \mathbf{I} \end{pmatrix} \in \mathbb{R}^{n_2+n_L,N}$, where $N = n_1 + n_2 + n_L = n + n_L$. The differential variable $\xi_p$ and the algebraic variable $\xi_q$ are given by

$$\begin{aligned} \xi_p &= \boldsymbol{p}^{*T}\boldsymbol{x} = \begin{pmatrix} \boldsymbol{p}_c^{*T}\boldsymbol{e} \\ \boldsymbol{\jmath}_L \end{pmatrix} \in \mathbb{R}^{n_2+n_L}, \\ \xi_q &= \boldsymbol{q}^{*T}\boldsymbol{x} = \boldsymbol{q}_c^{*T}\boldsymbol{e} \in \mathbb{R}^{n_1}. \end{aligned} \tag{10}$$

In [3], they decoupled index-1 systems using the above bases and their corresponding inverses but they also had to compute the inverse of $\mathbf{E}_1$. But inverting $\mathbf{E}_1$, might be computationally expensive and leads to very dense matrices of the decoupled system. In this paper, we try to avoid inverting $\mathbf{E}_1$ as follows: On additional to the above bases, we construct $\hat{\boldsymbol{p}}^T \in \mathbb{R}^{n_2+n_L,N}$, $\hat{\boldsymbol{q}}^T \in \mathbb{R}^{n_1,N}$ such that $\hat{\boldsymbol{p}}^T\mathbf{A}\boldsymbol{q} = 0$ and $\hat{\boldsymbol{q}}^T\mathbf{E}\boldsymbol{p} = 0$, given by $\hat{\boldsymbol{p}} \in \ker(\boldsymbol{q}_c^T\mathbf{A}_R\boldsymbol{G}\mathbf{A}_R^\top, -\boldsymbol{q}_c^T\mathbf{A}_L)$ and $\hat{\boldsymbol{q}} = \boldsymbol{q}$, since $\mathbf{E}$ is symmetric. Without loss of generality , index-1 system (4a) can be decoupled as

$$\overbrace{\hat{\boldsymbol{p}}^T\mathbf{E}\boldsymbol{p}}^{\mathbf{E}_p}\,\xi_p' = \overbrace{\hat{\boldsymbol{p}}^T\mathbf{A}\boldsymbol{p}}^{\mathbf{A}_p}\,\xi_p + \overbrace{\hat{p}^T\mathbf{B}}^{\mathbf{B}_p}\,u, \tag{11a}$$

$$\underbrace{-\hat{\boldsymbol{q}}^T\mathbf{A}\boldsymbol{q}}_{\mathbf{E}_q}\,\xi_q = \underbrace{\hat{\boldsymbol{q}}^T\mathbf{A}\boldsymbol{p}}_{\mathbf{A}_q}\,\xi_p + \underbrace{\hat{q}^T\mathbf{B}}_{\mathbf{B}_q}\,u, \tag{11b}$$

where (11a) and (11b) is the differential and algebraic parts. We can observe that there is no inversion of matrices, thus this decouple system is computationally cheaper to derive than it's counter part in [3] . If we use matrices in (1) the algebraic part (11b) can be written as

$$\boldsymbol{q}_c^T\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\boldsymbol{q}_c\xi_q = -\boldsymbol{q}_c^T\left(\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top \qquad \mathbf{A}_L\right)\boldsymbol{p}\xi_p - \boldsymbol{q}_c{}^T\mathbf{A}_I u. \tag{12}$$

The output equation (4b) can also be decomposed as

$$y = \mathbf{C}_p^T\xi_p + \mathbf{C}_q^T\xi_q, \tag{13}$$

where $\mathbf{C}_p = \boldsymbol{p}^T\mathbf{C}$ and $\mathbf{C}_q = \boldsymbol{q}^T\mathbf{C}$. If we assume $\mathbf{C} = \mathbf{B}$, thus we have $\mathbf{C}_p = \begin{bmatrix} -\boldsymbol{p}_c^T\mathbf{A}_I & 0 \end{bmatrix}$ and $\mathbf{C}_q = -\boldsymbol{q}_c^T\mathbf{A}_I$. Example below illustrates how to decouple RLC networks of the form (1) using the above proposed procedure.

*Example 1:* Consider an RLC circuit network with two current sources as shown in figure below. The incidence ma-

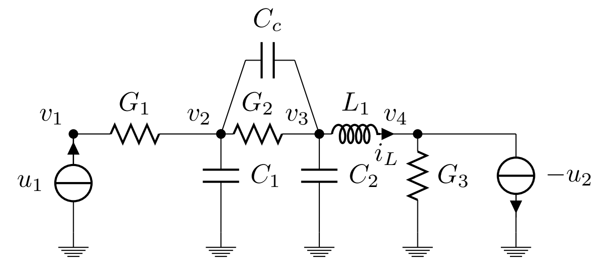


Fig. 1. Two port RLC circuit example

trices for capacitors , resistors , inductors and current sources are given by

$$\mathbf{A}_C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{A}_R = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbf{A}_L = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_I = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{14}$$

The capacitance, inductance and and conductance matrices are given by

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} C_c & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix}, \quad \boldsymbol{\mathcal{L}} = L_1 \quad \text{and} \quad \boldsymbol{\mathcal{G}} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix} \tag{15}$$

Substituting (14) and (15) into (1) we obtain a DAE with system matrices given by

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 + C_c & -C_c & 0 & 0 \\ 0 & -C_c & C_2 + C_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_1 \end{bmatrix},$$
$$\mathbf{A} = \begin{bmatrix} -G_1 & G_1 & 0 & 0 & 0 \\ G_1 & -G_1 - G_2 & G_2 & 0 & 0 \\ 0 & G_2 & -G_2 & 0 & -1 \\ 0 & 0 & 0 & -G_3 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \mathbf{B}, \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{16}$$

and the unknown $\boldsymbol{x} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{\jmath}_{L_1})^\top$. This system is solvable since $\det(\lambda\mathbf{E} - \mathbf{A}) \neq 0$ and it is of index -1 since the incidence matrices $\mathbf{A}_C$ and $\mathbf{A}_R$ satisfy condition (8). Following the procedure in the previous section we obtain the projectors $\mathbf{Q}_C$ and $\mathbf{P}_C$ given by

$$\mathbf{Q}_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The bases of $\mathbf{Q}_C, \mathbf{P}_C$ and their respective inverses are given by

$$\boldsymbol{q}_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{p}_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_c^{*T} = \boldsymbol{q}_c^T, \boldsymbol{p}_c^{*T} = \boldsymbol{p}_c^T. \tag{17}$$

Thus the bases of projectors $\mathbf{Q}_0$, $\mathbf{P}_0$ and their respective inverses are given by

$$\boldsymbol{q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{p} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}^{*T} = \boldsymbol{q}^T, \boldsymbol{p}^{*T} = \boldsymbol{p}^T. \tag{18}$$

Using (10) the differential and algebraic unknowns are given by

$$\xi_p = \boldsymbol{p}^{*T}\boldsymbol{x} = \begin{pmatrix} \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \\ \boldsymbol{\jmath}_{L_1} \end{pmatrix}, \quad \xi_q = \boldsymbol{q}^{*T}\boldsymbol{x} = \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_4 \end{pmatrix}.$$

The decoupling bases $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}^T$ are given by

$$\hat{\boldsymbol{p}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{G_3} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{q}} = \boldsymbol{q}. \tag{19}$$

Substituting (16) and (17)-(19) into (11), we obtain the decoupled system with coefficient matrices

$$\mathbf{E}_p = \begin{bmatrix} C_1 + C_c & -C_c & 0 \\ -C_c & C_2 + C_c & 0 \\ 0 & 0 & L \end{bmatrix}, \mathbf{A}_p = \begin{bmatrix} -G_2 & G_2 & 0 \\ G_2 & -G_2 & -1 \\ 0 & 1 & -\frac{1}{G_3} \end{bmatrix}$$
$$\mathbf{B}_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{G_3} \end{bmatrix}, \mathbf{E}_q = \begin{bmatrix} G_1 & 0 \\ 0 & G_3 \end{bmatrix}, \mathbf{A}_q = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbf{B}_q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C}_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{20}$$

### *B. RC network*

The RC network equation can easily be derived from (1) if we eliminate the the inductors in the RLC network leading to

$$\underbrace{\mathbf{A}_C\boldsymbol{\mathcal{C}}\mathbf{A}_C^\top}_{\mathbf{E}} \frac{d\boldsymbol{e}}{dt} = \underbrace{-\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top}_{\mathbf{A}} \boldsymbol{e} + \underbrace{-\mathbf{A}_I}_{\mathbf{B}} \boldsymbol{\imath}, \tag{21}$$

with consistent initial data $\boldsymbol{e}(0) = \boldsymbol{e}_0$,

where the node potentials $\boldsymbol{e}(t) \in \mathbb{R}^n$ are the only unknowns. Here, still $\boldsymbol{\mathcal{C}} \in \mathbb{R}^{n_C, n_C}$, and $\boldsymbol{G} \in \mathbb{R}^{n_G, n_G}$ are the capacitance and conductance matrices, which are assumed to be symmetric and positive-definite. This system can also be written in the form (4), we can assume $\mathbf{C} = \mathbf{B}$. Assuming the matrix pencil $(\mathbf{E}, \mathbf{A})$ is regular, them we can also compute its tractability index as follows: Setting $\mathbf{E}_0 = \mathbf{E}$ and $\mathbf{A}_0 = \mathbf{A}$ leads to

$$\mathbf{E}_0 = \mathbf{A}_C\boldsymbol{\mathcal{C}}\mathbf{A}_C^\top,\ \mathbf{A}_0 = -\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top,\ \mathbf{B} = -\mathbf{A}_I \quad \text{and}\ \boldsymbol{u} = \boldsymbol{\imath}.$$

We denote by $\mathbf{Q}_0 = \mathbf{Q}_C$ the projector onto the kernel of $\mathbf{A}_C^\top$, and set $\mathbf{P}_0 = \mathbf{P}_C = \mathbf{I} - \mathbf{Q}_C$, such that $\mathbf{P}_C\mathbf{Q}_C = \mathbf{Q}_C\mathbf{P}_C = \mathbf{0}$. Then, we can find

$$\mathbf{E}_1 = \mathbf{E}_0 - \mathbf{A}_0\mathbf{Q}_0 = \mathbf{A}_C\boldsymbol{\mathcal{C}}\mathbf{A}_C^\top + \mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\mathbf{Q}_C,$$
$$\mathbf{A}_1 = \mathbf{A}_0\mathbf{P}_0 = \mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\mathbf{P}_C.$$

It is also easy to show that If, we have the conditions:

$$\mathrm{Ker}(\mathbf{A}_C, \mathbf{A}_R)^\top = \{0\}, \tag{22}$$

then we find that $\boldsymbol{e} \in \ker \mathbf{E}_1$ if and only if $\mathbf{Q}_C\boldsymbol{e} = 0$. Thus the condition (22) is equivalent to the index-1 condition $\det \mathbf{E}_1 \neq 0$ as for the case for RLC networks with only current sources.

*1) Index-1 RC network:* In this case we construct bases for projectors $\mathbf{Q}_0 = \mathbf{Q}_C$ and $\mathbf{P}_0 = \mathbf{P}_C$ given by $\boldsymbol{q} = \boldsymbol{q}_c \in \mathbb{R}^{n, n_1}$, $\boldsymbol{p} = \boldsymbol{p}_c \in \mathbb{R}^{n, n_2}$ and their respective inverses is given by $\boldsymbol{q}^{*T} = \boldsymbol{q}_c^{*T} \in \mathbb{R}^{n_1, n}$, $\boldsymbol{p}^{*T} = \boldsymbol{p}_c^{*T} \in \mathbb{R}^{n_2, n}$, where $n = n_1 + n_2$. The differential variable $\xi_p$ and the algebraic variable $\xi_Q$ are given by

$$\xi_p = \boldsymbol{p}^{*T}\boldsymbol{x} = \boldsymbol{p}_c^{*T}\boldsymbol{e} \in \mathbb{R}^{n_2}, \quad \xi_q = \boldsymbol{q}^{*T}\boldsymbol{x} = \boldsymbol{q}_c^{*T}\boldsymbol{e} \in \mathbb{R}^{n_1}.$$

In order to decouple the DAE system, we construct $\hat{\boldsymbol{p}}^T \in \mathbb{R}^{n_2, n}$, $\hat{\boldsymbol{q}}^T \in \mathbb{R}^{n_1, n}$ such that $\hat{\boldsymbol{p}}_0^T\mathbf{A}\boldsymbol{q}_0 = 0$ and $\hat{\boldsymbol{q}}_0^T\mathbf{E}\boldsymbol{p}_0 = 0$, given by $\hat{\boldsymbol{p}} \in \ker(\boldsymbol{q}_c^T\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top)$ and $\hat{\boldsymbol{q}} = \boldsymbol{q}$ since $\mathbf{E}$ is symmetric. From (11) and (13), we can decouple index-1 RC network as

$$\mathbf{E}_p\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p u, \tag{23a}$$
$$\mathbf{E}_q\xi_q = \mathbf{A}_q\xi_p + \mathbf{B}_q u, \tag{23b}$$
$$y = \mathbf{C}_p^T\xi_p + \mathbf{C}_q^T\xi_q, \tag{23c}$$

where $\mathbf{E}_p = \hat{\boldsymbol{p}}^T\mathbf{A}_C\boldsymbol{\mathcal{C}}\mathbf{A}_C^\top\boldsymbol{p}$, $\mathbf{A}_p = -\hat{\boldsymbol{p}}^T\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\boldsymbol{p}$, $\mathbf{B}_p = -\hat{p}^T\mathbf{A}_I$, $\mathbf{E}_q = \hat{\boldsymbol{q}}^T\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\boldsymbol{q}$, $\mathbf{A}_q = -\hat{\boldsymbol{q}}^T\mathbf{A}_R\boldsymbol{\mathcal{G}}\mathbf{A}_R^\top\boldsymbol{p}$, $\mathbf{B}_q = -\hat{q}^T\mathbf{A}_I$, and $\mathbf{C}_p^T = -\boldsymbol{p}_c^T\mathbf{A}_I$, $\mathbf{C}_q^T = -\boldsymbol{p}_c^T\mathbf{A}_I$. (23a) and (23b) is the differential and algebraic parts. We note that the $\mathbf{E}_p$ and $\mathbf{E}_q$ must always be non-singular for any index-1 system.

## IV. IMPLICIT-IMOR METHOD

In this section, we propose the Implicit -index-aware model order reduction (Implict-IMOR) method which is the modification of the index-aware model order reduction method proposed in [3], [2]. In this paper, we shall call this method the explicit-IMOR method. In the explicit-IMOR method we apply the reduction on the explicit decoupled system while the implicit-IMOR method we apply it on the implicit decoupled system (23). System (23) can be written in the form (4) given by

$$\tilde{\mathbf{E}}\xi' = \tilde{\mathbf{A}}\xi + \tilde{\mathbf{B}}u \tag{24a}$$
$$\boldsymbol{y} = \tilde{\mathbf{C}}^T\xi, \tag{24b}$$

where $\tilde{\mathbf{E}} = \begin{bmatrix} \mathbf{E}_p & 0 \\ 0 & 0 \end{bmatrix}, \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_p & 0 \\ \mathbf{A}_q & -\mathbf{E}_q \end{bmatrix} \in \mathbb{R}^{N,N}$, $\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_p \\ \mathbf{B}_q \end{bmatrix} \in \mathbb{R}^{N,m}$, $\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_p \\ \mathbf{C}_q \end{bmatrix} \in \mathbb{R}^{N,\ell}$ and the projected state space $\xi = \begin{pmatrix} \xi_p^T & \xi_q^T \end{pmatrix}^T \in \mathbb{R}^N$, where $\xi_p \in \mathbb{R}^{n_p}, \xi_q \in \mathbb{R}^{n_q}$ and $N = n_p + n_q$. We note it can easily be proved that system (24) and (4) are equivalent for index-1 systems. Moreover it can be proved that the finite spectrum of the matrix pencil $(\mathbf{E}, \mathbf{A})$ is equal to the spectrum of $(\mathbf{E}_p, \mathbf{A}_p)$. Thus the decoupling procedure preserves the spectrum of the original system. The transfer function of the two system also coincides, that is $\mathbf{H}(s) = \hat{\mathbf{H}}(s) = \tilde{\mathbf{C}}^T(s\tilde{\mathbf{E}} - \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{B}}$. The transfer function $\hat{\mathbf{H}}(s)$ can also be decomposed as

$$\hat{\mathbf{H}}(s) = \mathbf{H}_p(s) + \mathbf{H}_q(s), \tag{25}$$

where $\mathbf{H}_p(s) = \mathbf{C}_p^T(s\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{B}_p$ and $\mathbf{H}_q(s) = \mathbf{C}_q\mathbf{E}_q^{-1}\left[\mathbf{A}_q(s\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{B}_p + \mathbf{B}_q\right]$. Strictly separating (24), we obtain a differential and algebraic subsystems given by

$$\mathbf{E}_p\xi_p' = \mathbf{A}_p\xi_p + \mathbf{B}_p u, \tag{26a}$$

$$\boldsymbol{y}_p = \mathbf{C}_p^T\xi_p, \tag{26b}$$

and

$$\mathbf{E}_q\xi_q = \mathbf{A}_q\xi_p + \mathbf{B}_q u, \tag{27a}$$

$$\boldsymbol{y}_q = \mathbf{C}_q^T\xi_q. \tag{27b}$$

The output solution can be obtained using $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_q$. We can observe that $\mathbf{H}_p(s)$ and $\mathbf{H}_q(s)$ are the transfer functions of (26) and (27), respectively.

The differential subsystem (26) can be reduced using the model order reduction methods for ODEs such as the PRIMA method. Thus we can approximate $\xi_p = \mathbf{V}_p\xi_{p_r}$, where $\mathbf{V}_p \in \mathbb{R}^{n_p, r_1 m}$ is an orthonormal basis matrix for $\mathcal{K}_{r_1}(\mathbf{M}_p, \mathbf{R}_p) = \text{span}\{\mathbf{R}_p, \mathbf{M}_p\mathbf{R}_p, \cdots, \mathbf{M}_p^{r_1-1}\mathbf{R}_p\}, r_1 \leq n_p$, where $\mathbf{M}_p = (s_0\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{E}_p$ and $\mathbf{R}_p = (s_0\mathbf{E}_p - \mathbf{A}_p)^{-1}\mathbf{B}_p$. Then the reduced-order ODE subsystem is given by

$$\mathbf{E}_{p_r}\xi_{p_r}' = \mathbf{A}_{p_r}\xi_{p_r} + \mathbf{B}_{p_r}u, \tag{28a}$$

$$\boldsymbol{y}_{p_r} = \mathbf{C}_{p_r}^T\xi_{p_r}, \tag{28b}$$

where $\mathbf{E}_{p_r} = \mathbf{V}_p^T\mathbf{E}_p\mathbf{V}_p, \mathbf{A}_{p_r} = \mathbf{V}_p^T\mathbf{A}_p\mathbf{V}_p \in \mathbb{R}^{r_1,r_1}, \mathbf{B}_{p_r} = \mathbf{V}_p^T\mathbf{B}_p \in \mathbb{R}^{r_1,m}$ and $\mathbf{C}_{p_r} = \mathbf{V}_p^T\mathbf{C}_p \in \mathbb{R}^{r_1,\ell}$.

We observe that the above reduction of the differential part induces a reduction in the algebraic part (27a) but its dimension is unchanged given by

$$\mathbf{E}_q\xi_q = \mathbf{A}_q\mathbf{V}_p\xi_{p_r} + \mathbf{B}_q u. \tag{29}$$

We have already seen that the differential variable $\xi_p$ is confined to the subspace $\mathcal{V}_p = \mathcal{K}_{r_1}(\mathbf{M}_p, \mathbf{R}_p)$ spanned by $\mathbf{V}_p$. Then from (29), we can observe that the algebraic variable $\xi_q$ belongs to the subspace $\mathcal{V}_q := \text{span}(\mathbf{E}_q^{-1}\mathcal{W}_q)$ in $\mathbb{R}^{n_q}$, where $\mathcal{W}_q := \text{span}(\mathbf{B}_q, \mathbf{A}_q\mathbf{V}_p)$. Then $\mathcal{V}_q$ and $\mathcal{W}_q$ are spanned by $\mathbf{V}_q = \text{Orth}(\mathcal{V}_q)$ and $\mathbf{W}_q = \text{Orth}(\mathcal{W}_q)$, respectively. We note that $\mathbf{V}_q$ and $\mathbf{W}_q$ must be of the same dimension. Thus substituting $\xi_p = \mathbf{V}_p\xi_{p_r}$ and $\xi_q = \mathbf{V}_q\xi_{q_r}$ into (27) and left multiplying (27a) by $\mathbf{W}_q^T$ , we obtain the reduced-order model of the algebraic part (27) given by

$$\mathbf{E}_{q_r}\xi_{q_r} = \mathbf{A}_{q_r}\xi_{p_r} + \mathbf{B}_{q_r}u, \tag{30a}$$

$$\boldsymbol{y}_{q_r} = \mathbf{C}_q^T\xi_{q_r}. \tag{30b}$$

where $\mathbf{E}_{q_r} = \mathbf{W}_q^T\mathbf{E}_q\mathbf{V}_q \in \mathbb{R}^{r_2,r_2}$, $\mathbf{A}_{q_r} = \mathbf{W}_q^T\mathbf{A}_q\mathbf{V}_p \in \mathbb{R}^{r_2,r_1}$, $\mathbf{B}_{q_r} = \mathbf{W}_q^T\mathbf{B}_q \in \mathbb{R}^{r_2,m}$ and $\mathbf{C}_{q_r} = \mathbf{V}_q^T\mathbf{C}_q \in \mathbb{R}^{r_2,\ell}$, where $r_1$ is the reduced dimension of the differential part and $r_2 = \dim(\mathcal{W}_q) = \dim(\mathcal{V}_q)$ which is equal to the reduced dimension of the algebraic part. Hence recombining (28) and (30) we obtain the implicit-IMOR reduced-order model for (3) given by

$$\tilde{\mathbf{E}}_r\xi_r' = \tilde{\mathbf{A}}_r\xi_r + \tilde{\mathbf{B}}_r u \tag{31a}$$

$$\boldsymbol{y}_r = \tilde{\mathbf{C}}_r^T\xi_r, \tag{31b}$$

where $\tilde{\mathbf{E}}_r = \tilde{\mathbf{W}}^T\tilde{\mathbf{E}}\tilde{\mathbf{V}}$, $\tilde{\mathbf{A}}_r = \tilde{\mathbf{W}}^T\tilde{\mathbf{A}}\tilde{\mathbf{V}}$, $\tilde{\mathbf{B}}_r = \tilde{\mathbf{W}}^T\tilde{\mathbf{B}}, \tilde{\mathbf{C}}_r = \tilde{\mathbf{V}}^T\tilde{\mathbf{C}}$, where $\tilde{\mathbf{W}} = \begin{bmatrix} \mathbf{V}_p & 0 \\ 0 & \mathbf{W}_q \end{bmatrix}, \tilde{\mathbf{V}} = \begin{bmatrix} \mathbf{V}_p & 0 \\ 0 & \mathbf{V}_q \end{bmatrix}$. We note that it can easily be proved that the implicit-IMOR methods also preserves the goals of the MOR depending on the MOR method used to reduce the differential part and moreover it preserve the index of the DAE. For instance if PRIMA method is used to reduce the differential part then the passivity can be guaranteed if the original matrices satisfies properties suggested in [7]. We note the projectors and their corresponding bases are numerically feasible and can be computed using the LU based routine proposed in [9] for the case of sparse matrices. But for the case of dense matrices SVD based routines have to be used.

## V. Numerical experiments

In Tab I, we decouple Electric Power Grids models. These are real world index-1 DAE models which can be downloaded from [12]. In Tab I, $n_p$ and $n_q$ represents the number of differential and algebraic equations, respectively. We can observe that $N = n_p + n_q$, thus the dimension of the DAE system is preserved. $m$ and $\ell$ is the number of inputs and outputs respectively. In the Tab II, we compared the computational cost of splitting the power systems using the implicit and explicit decoupling procedure. We can observe that the proposed implicit decoupling procedure is computationally far cheaper than the explicit decoupling procedure proposed in [3].

TABLE I. Dimension of decoupled Electric Power Grids index-1 DAE models

| Power system | Decoupled dimension | | Dimension | #inputs | #outputs |
|---|---|---|---|---|---|
| | $n_p$ | $n_q$ | $N$ | $m$ | $\ell$ |
| Juba5723 | 5723 | 34614 | 40337 | 2 | 1 |
| Bauru5727 | 5727 | 34639 | 40366 | 2 | 2 |
| xingo3012 | 3012 | 17932 | 20944 | 2 | 2 |
| BIPS/1997 | 1664 | 11586 | 13250 | 1 | 1 |

TABLE II. Comparison of the computational costs of the splitting methods.

| Power system | Implicit splitting method | Explicit splitting method |
|---|---|---|
| Juba5723 | 40.50Seconds | 1466.93Seconds |
| Bauru5727 | 42.88Seconds | 1568.32Seconds |
| xingo3012 | 16.64Seconds | 102.88Seconds |
| BIPS/1997 | 3.78Seconds | 39.95Seconds |

In the Tab III, we compared the sparsity of the matrix pencil of the implicit and explicit decoupled system in descriptor form (24). In Tab III, we observe that matrix $\tilde{\mathbf{A}}$ using implicit splitting is sparser than that of the explicit splitting method, while matrix $\tilde{\mathbf{E}}$ the viceversa is true. This is due to

TABLE III. COMPARISON OF THE NUMBER OF NONZEROS (NNZ) OF $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$.

| Power system | Implicit splitting method | | Explicit splitting method | |
|---|---|---|---|---|
| | nnz($\tilde{\mathbf{E}}$) | nnz($\tilde{\mathbf{A}}$) | nnz($\tilde{\mathbf{E}}$) | nnz($\tilde{\mathbf{A}}$) |
| Juba5723 | 1807062 | 352416 | 5723 | 21993176 |
| Bauru5727 | 1810871 | 3527592 | 5727 | 22003930 |
| xingo3012 | 503925 | 1004277 | 3012 | 5833194 |
| BIPS/1997 | 201542 | 448799 | 1664 | 2611782 |

the fact that $\tilde{\mathbf{A}}$ does not involve matrix inversion for the case of implicit splitting method. Explicit splitting method has a sparser matrix $\tilde{\mathbf{E}}$ just because $\mathbf{E}_p$ and $\mathbf{E}_q$ are always identity matrices. Another advantage of implicit splitting over explicit splitting method that it partially preserves the original structure of matrix pencil $(\mathbf{E}, \mathbf{A})$ which is very important in the electric networks community.

For convenience we use the last example in Tab I to compare the reduced-order models obtained using the IIMOR and IMOR methods which is a SISO model but the IIMOR method can as well be used on the MIMO models. Using $s_0 = 10$ as the expansion point, we were able to reduce the differential and algebraic part of decoupled system to 375 and 100, respectively using the IIMOR and IMOR method. Thus the dimension of the reduced-order models is 475. In Fig. 2, we compare the magnitude of the transfer function of the IIMOR and IMOR reduced -order models. We observe that both reduced-order model coincides with that of the original models. In Fig. 3, we compare the their respective approximation error. We can see that IMOR reduced-order model is more accurate than the IIMOR method. However, even if the IMOR method may be more accurate than the IIMOR method it is computationally expensive. Therefore, we need to trade off between accuracy and computational costs while using the two methods.

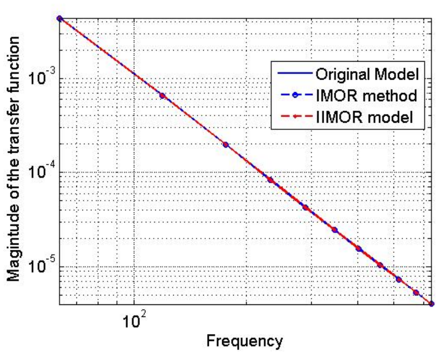


Fig. 2. Comparison of Transfer function

## VI. CONCLUSION

In conclusion, we have have proposed the IIMOR method which an implicit version of the IMOR method. This method is computationally cheaper than the IMOR method. Also it partially preserves the original structure of the DAE system. The IMOR method may be more accurate than the IIMOR method but it is computationally expensive to be use to reduce large RC/RLC networks. Hence IIMOR method is the best optional. Finally, the IIMOR method can be extended to systems with higher tractability index. This will be the topic of a forthcoming paper.

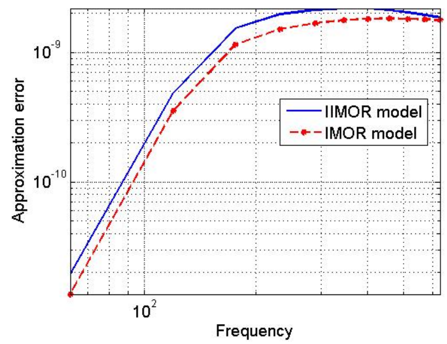


Fig. 3. Comparison of approximation error.

## ACKNOWLEDGMENT


This work was funded by The Netherlands Organisation for Scientific Research (NWO).


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