Towards the Formal Analysis of Microresonators based Photonic Systems

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***Abstract*—Recent developments in the fabrication technology attracted the attention of optical engineers and physicists in the area of VLSI photonics. Due to the physical nature of light-wave systems and their usage in safety critical domains such as human surgeries and high budget space missions, it is indispensable to build high assurance systems. Traditionally, the analysis of such systems has been carried out by paper-and-pencil based proofs and numerical computations. However, these techniques cannot provide perfectly accurate results due to the risk of human error and inherent approximations of numerical algorithms. In order to overcome these limitations, we propose to use higher-order logic theorem proving to improve the analysis in the domain of integrated optics or VLSI photonics. In particular, this paper provides a higher-order logic formalization of optical microresonators which are the most fundamental building blocks of many photonic devices. In order to illustrate the practical utilization of our work, we present the formal analysis of 2-D microresonator lattice optical filters.**

I. INTRODUCTION

In the past few decades, optics and photonics technology has emerged as a promising solution to resolve many bottlenecks (e.g., high capacity telecommunication links with smaller size) in traditional semiconductor technology. As a result, optics and photonics devices are increasingly used in telecommunication, human surgeries, energy and environmental management systems. The main advantages of optics are high-speed, low power, huge bandwidth, smaller size and fast information processing. Nowadays, communication systems are composed of both optical and electronic devices such as long distance communication links, which are a combination of optical fiber and electrically controlled optical switches. In order to meet future challenges, significant research has been done in the area of developing very large scale integrated (VLSI) photonic circuits which are the optical counter parts of traditional VLSI electronic chips [5]. An optical microresonator [12] is the most fundamental building block of VLSI photonics and it is an integral part of many devices such as optical filters, optical switches, optical transistors, wavelength division multiplexing and biosensors. Optical microresonators confine light in a closed structure by the process of total internal reflection to achieve desired functionalities such as light amplification and frequency selection. Due to their small size and flexible geometry, such resonators are preferred over Fabry-Pérot resonators [17] in the design of integrated optics.

The development of optical integrated circuits mainly involves the physical modeling of some fundamental building-blocks such as microresonators and couplers, which are used to confine light and transfer energy between two waveguides, respectively [19]. One of the most critical requirement is the validation of such models and the verification of system properties. Therefore, significant portion of time is spent to find bugs in the design process prior to the manufacturing of the actual system. Traditionally, the models of these building-blocks are constructed using paper-and-pencil equations by optical engineers and physicists. One of the primary but most time consuming analytical approach to analyze the properties of microresonator circuits is to explicitly write the node and loop equations and then computing complex output wave amplitudes normalized by the input amplitude. The main strength of this technique is that it provides almost all important scattering coefficients such as transfer intensity, phase and dispersion [4]. Another approach is the use of transfer matrices to characterize different types of optical circuits [18]. This method provides an easy way to model complex optical systems and their analysis using complex linear algebra. Although, these analytical methods provide closed form mathematical solutions but carrying such an analysis by-hand is human error-prone, particularly for systems involving many optical components. Moreover, most of the underlying assumptions are not specified explicitly which may lead to faulty system designs. There are many examples of erroneous analysis in optics literature, but a recent one can be found in [3] and its identification and correction is reported in [16].

Recently, high-speed computing resources are actively used to perform simulation based analysis using numerical algorithms. The most commonly used numerical techniques are finite-difference time-domain (FDTD) modeling of electromagnetic equations [22] and the transmission line modeling (TLM) method [2]. Both of these methods have been proven to be very time consuming in most optics and photonics problems such as optical waveguide structures and optical fibres [19]. Since optical microresonators trap light for a long time, the simulation time should be extremely large in order to achieve reasonable results [20]. Besides the huge memory and computational time requirements, these techniques cannot provide perfectly accurate results due to the discritization of continuous parameters and the involvement of unverified numerical algorithms. The above mentioned inaccuracy problems of traditional analysis techniques are impeding their usage in designing safety-critical optical systems, where minor bugs can lead to disastrous consequences such as the loss of human lives or financial loss because of their use in high budget defense and space missionss.

In order to address similar inaccuracy problems in electronic devices, many formal and semi-formal verification techniques have been proposed. Recently, some preliminary works for analyzing optical systems using theorem proving

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[9] have been reported in the open literature. For instance, in [11], the formal analysis of optical waveguides using HOL4 theorem prover is reported. This work is primarily based on real analysis which is insufficient to capture the dynamics of most optical and photonic systems. For example, electric and magnetic fields can only be modeled using complex vectors theory, which to the best of our knowledge is not available in the HOL4 theorem prover. In [21], the authors developed a preliminary infrastructure in HOL Light theorem prover [8] to formally analyze optical systems based on ray optics. The developed infrastructure is only applicable where the size of the optical components is much larger than the wavelength of light (which is assumed to be very small). Note also that ray optics can only be applied to analyze some basic properties of optical systems such as resonators stability [17]. Despite of the vast applications of VLSI photonics in safety and mission critical applications, none of the above mentioned work provides the basis (i.e., formalization of basic building-blocks such as microresonators and interference couplers [18]) to apply formal verification in this domain.

The main focus of this paper is to bridge the above mentioned gap and strengthen the formal reasoning support in the area of integrated optics. Our main goal is to develop a higher-order logic formalization of most widely used building-blocks involved in the design of practical photonic systems. In this paper, we build upon the rich multivariate analysis libraries [10] of the HOL Light theorem prover along with the complex matrices formalization which are the foremost requirements to model the physical dynamics of such building-blocks. As a first step towards our ultimate goal, we present in this paper the higher-order logic formalization of optical microresonators. We provide a set of formal definitions to model most commonly used microresonators structures, i.e., resonator coupled with one waveguide and two waveguides. We derive the transfer matrices of each resonator structure which provides the basis to model real-world photonic circuits. In order to reason about periodic optical structures, we present the formal verification of Sylvester's theorem [23]. In order to show the practical utilization of our work, we present the formal analysis of 2-D microresonator lattice optical filter by decomposing into two 1-D linear cascades of coupled and uncoupled microresonators, respectively. To the best of our knowledge, this is the first time that formal methods has been used in the area of VLSI photonics. Moreover, we have been able to find some discrepancies in the paper and pencil based proof approach [14]. The most important one is the identification of a missing assumption in Sylvester's theorem which plays a central role in the analysis of optical filters.

II. Optical Microresonators

Optical microresonators[1], also named as microring resonators (MRR) [7], are optical structures made of different reflecting surfaces to confine the light in very small volumes to perform different operations such as light amplification and wavelength filtering. A single microring resonator can be characterized by its reflectivity (r), transmissivity (t), cavity length (L_c), power attenuation (α), wavelength λ, and effective waveguide index (n_{eff}) as shown in Figure 1.

[1]Throughout this paper, microresonators refer to microring resonators [19] which are different from Fabry-Pérot cavity [17] based microresonators.

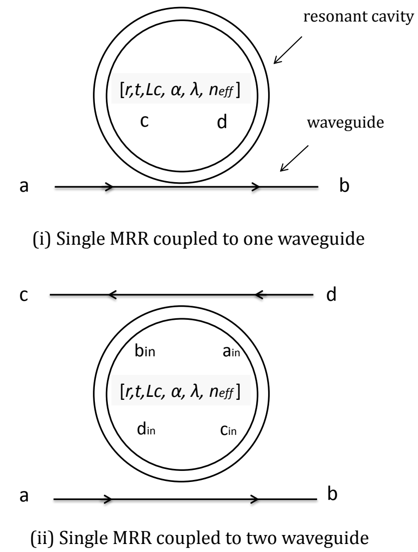

Fig. 1. Schematic Structure of Optical Microresonators (MRR)

In order to facilitate the formal reasoning process, we represent a microring resonators as a new type definition in HOL Light [2] as follows:

***Definition** 1 (Microring Resonator (MRR)):*

```
new_type_abbrev "mrr =
                 :ℝ × ℝ × ℝ × ℝ× ℝ × ℝ"
```

Here, the type `mrr` is a composition of six real numbers (r,t,L_c,α,λ,n_{eff}) which are necessary parameters to model a single resonator as described above.

In practice, MRRs works as either a two or a four port configuration in integrated optics. In a two-port resonator structure (see Figure 1(i)), the resonant cavity is coupled to a single waveguide and hence it has only a single input a and output b. Note that a and b are complex-valued parameters, which essentially represent electromagnetic fields at the input and output, whereas c and d represent fields inside the cavity. Such resonators are suitable as a dispersive or attenuating element and has been widely used in phase-only filters [6]. If a single ring is coupled to two bus waveguides then the configuration is a four-port as shown in Figure 1(ii), where a,b,c,d are the input, throughput, dropped and the added fields, respectively. We define a resonator structure by a new type in HOL Light as follows:

***Definition** 2 (Microring Resonator Structure):*

```
define_type "mrr_structure =
              two_port | four_port"
```

Next, we define what is the valid behavior of a MRR in terms of the relation between resonator parameters (r,t,L_c,α,λ,n_{eff}) and field parameters (a,b,c,d) at the input and the output. For the one-port MRR (Figure 1 (i)), it is necessary to explicitly define the relation between fields inside and outside of the resonator. On the other hand, it is sufficient to model the physical behavior using the two input and two

[2]Note that throughout in this paper, we used minimal HOL Light syntax in the presentation of definitions and theorems to improve the readability and for the better understanding without prior experience of HOL Light.

output fields in case of four-port MRR (Figure 1 (ii)) [14]. Then the predicate is defined by case analysis on the MRR structure:

***Definition** 3 (Valid Behavior in MRR Structures):*

$\vdash$ (**is_valid_behavior_in_mrr** (a,d) (b,c) $(\mathtt{r},\mathtt{t},\mathtt{L_c},\alpha,\lambda,\mathtt{n_{eff}})$:mrr four_port $\Leftrightarrow$
let $\delta = (\frac{2*\pi}{\lambda}) * \mathtt{n_{eff}} * \mathtt{L_c}$ and $\tau = \mathtt{exp}(-\frac{\alpha * \mathtt{Lc}}{2})$ in
let $\mathtt{R} = -\frac{\mathtt{r}*(1-\tau*\mathtt{exp}(-\mathtt{j}*\delta))}{1-\mathtt{r}^2*\tau*\mathtt{exp}(-\mathtt{j}*\delta)}$ and
$\mathtt{T} = -\frac{\mathtt{t}^2*\sqrt{(\tau)}*\mathtt{exp}(-\frac{\mathtt{j}*\delta}{2})}{1-\mathtt{r}^2*\tau*\mathtt{exp}(-\mathtt{j}*\delta)}$ in
$\mathtt{d} = \frac{1}{\mathtt{R}} * \mathtt{c} - \frac{\mathtt{T}}{\mathtt{R}} * \mathtt{a} \wedge \mathtt{b} = \frac{\mathtt{T}}{\mathtt{R}} * \mathtt{c} + \frac{\mathtt{R}^2 - \mathtt{T}^2}{\mathtt{R}} * \mathtt{a}) \wedge$
(**is_valid_behavior_in_mrr** (a,d) (b,c) $(\mathtt{r},\mathtt{t},\mathtt{L_c},\alpha,\lambda,\mathtt{n_{eff}})$:mrr two_port $\Leftrightarrow$
$\mathtt{c} = -\frac{1}{\mathtt{j}*\mathtt{t}} * (\mathtt{a} + \mathtt{r} * \mathtt{b}) \wedge \mathtt{d} = \frac{1}{\mathtt{j}*\mathtt{t}} * (\mathtt{r} * \mathtt{a} + \mathtt{b}))$

Here, is_valid_behavior_in_mrr takes four fields parameters $(\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d} \in \mathbb{C})$, a microring resonator $(\mathtt{r},\mathtt{t},\mathtt{L_c},\alpha,\lambda,\mathtt{n_{eff}})$ and mrr_structure, and returns the relation among these parameters. Note that j represents an imaginary unit and $\mathtt{j}^2 = -1$. The parameter δ represents the frequency-dependent phase shift, τ represents the waveguide loss effect, T and R represent the output field in the backward direction and forward direction, respectively.

The transfer matrix modeling [18] is the most widely used approach to analytically model MRRs [2]. The main characteristics of this technique are to decompose photonic circuits in the form of series of MRRs and then analyzing different behaviors using complex matrix algebra. Now, equipped with the above formal definitions (Definitions 1-3), we verify the transfer matrix relation of MRRs in case of two-port and four-port structures [14].

***Theorem** 1 (MRR Matrix for Two-Port Structure):*

$\vdash \forall$a b c d **r t** $\mathtt{L_c}$ α λ $\mathtt{n_{eff}}$.
is_valid_behavior_in_mrr (a,d) (b,c) $(\mathtt{r},\mathtt{t},\mathtt{L_c},\alpha,\lambda,\mathtt{n_{eff}})$:mrr two_port $\Longrightarrow$

$$\begin{bmatrix} \mathtt{c} \\ \mathtt{d} \end{bmatrix} = \frac{1}{\mathtt{j}*\mathtt{t}} \begin{bmatrix} -1 & -\mathtt{r} \\ \mathtt{r} & 1 \end{bmatrix} ** \begin{bmatrix} \mathtt{a} \\ \mathtt{b} \end{bmatrix}$$

***Theorem** 2 (MRR Matrix for Four-Port Structure):*

$\vdash \forall$a b c d **r t** $\mathtt{L_c}$ α λ $\mathtt{n_{eff}}$.
is_valid_behavior_in_mrr (a,d) (b,c) $(\mathtt{r},\mathtt{t},\mathtt{L_c},\alpha,\lambda,\mathtt{n_{eff}})$:mrr four_port $\Longrightarrow$
let $\delta = (\frac{2*\pi}{\lambda}) * \mathtt{n_{eff}} * \mathtt{L_c}$ and $\tau = \mathtt{exp}(-\frac{\alpha * \mathtt{Lc}}{2})$ in
let $\mathtt{R} = -\frac{\mathtt{r}*(1-\tau*\mathtt{exp}(-\mathtt{j}*\delta))}{1-\mathtt{r}^2*\tau*\mathtt{exp}(-\mathtt{j}*\delta)}$ and
$\mathtt{T} = -\frac{\mathtt{t}^2*\sqrt{(\tau)}*\mathtt{exp}(-\frac{\mathtt{j}*\delta}{2})}{1-\mathtt{r}^2*\tau*\mathtt{exp}(-\mathtt{j}*\delta)}$ in

$$\begin{bmatrix} \mathtt{d} \\ \mathtt{b} \end{bmatrix} = \frac{1}{\mathtt{R}} \begin{bmatrix} 1 & -\mathtt{T} \\ \mathtt{T} & \mathtt{R}^2 - \mathtt{T}^2 \end{bmatrix} ** \begin{bmatrix} \mathtt{c} \\ \mathtt{a} \end{bmatrix}$$

Here, $**$ represents the matrix-vector multiplication in HOL Light. The verification of above theroems mainly involves the rewriting of predicate defintions along with the properties of complex matrices.

III. Formal Analysis of 2-D Microresonator Lattice Photonic Filters

Photonic filters are widely used devices to selectively reject or transmit a range of wavelengths. The main applications of optical filters are in the area of spectroscopy, biochemical analysis, wavelength division (WDM) multiplexing and aerospace. One of the central element in such filters is resonant structure. An optical mirroring resonator can be used as a building-block for different optical filters [13]. In general, there are two types of configurations to build these filters: in the first configuration (also called Type I), the resonators are not mutually coupled but are periodically coupled to two side waveguides, with equal spacing between adjacent resonators. In the second configuration (also called Type II), the rings are mutually coupled in a linear cascade which is coupled to input and output bus waveguides [14]. In both types, the bandpass response of the filter is better than a single MRR, but there are several drawbacks (for example, ripples and sidelobes).

In this paper, we consider an alternative configuration presented in [14], that is a geometric hybrid of the Type I and Type II configurations . The configuration is a two-dimensional (2-D) filter that can be constructed as a periodically coupled array of coupled ring filters. The overall structure $(M \times N)$ consists of N independent columns of microring resonators side-coupled to two bus waveguides, with an equal spacing between columns and each column consisting of M coupled resonators as shown in Figure 2.

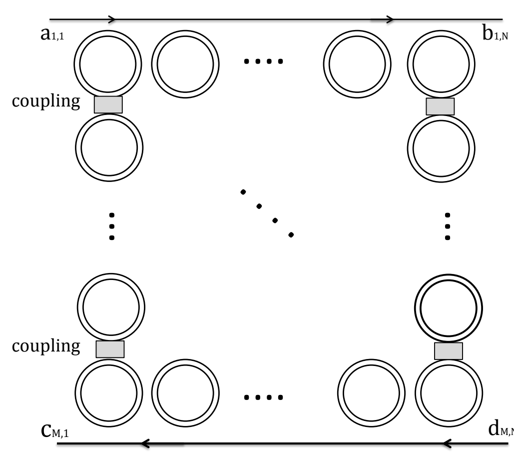

Fig. 2. 2-D Lattice of MRRs with M rows and N columns coupled to two parallel waveguids

One of the most important design criterion of photonic filters is the output response in terms of *transmissivity* and *reflectivity*, which describe the transmission and reflection intensity of light at the output of the filter, respectively [14]. The analysis of the output response of 2-D lattice (Figure 2) seems non-trivial at first because the number of columns N and rows M can be very large. Intuitively, a 2-D lattice can be decomposed into a row sublattice of uncoupled resonators and a column sublattice of coupled resonators, as shown in Figure 3. Then it is possible to analyze the properties of the 2-D filter in terms of the properties of the 1-D filters [14]. Next, we use our formalization of MRRs developed in the previous section to formally verify the response of 1-D filters consisting of an array of N microring resonators.

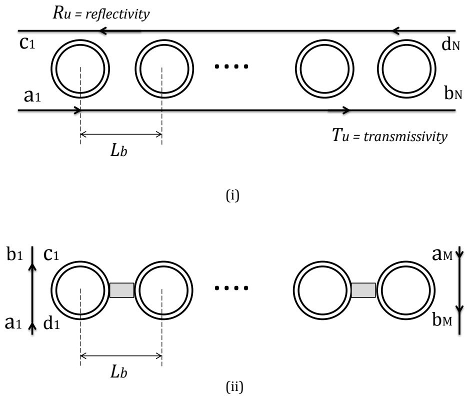

Fig. 3. Decomposition of 2-D Lattice (i) a row sublattice consisting of a linear cascade of N uncoupled resonators between the double waveguide (ii) a column sublattice consisting of a linear cascade of M mutually coupled resonators between the waveguides

We consider a linear cascade of MRRs periodically coupled to a pair of bus waveguides with constant spacing (L_b), as shown in Figure 3 (i). Generally, each MRR can be different from the other in terms of its parameters depending upon the filter specifications. The first step towards the formal verification of the given filter structure is to define the necessary notions in HOL Light. For example, we need to define the notion of a cascade of microring resonators and the input and output fields associated to each resonator in a cascade. The next step is to formally describe the physical behavior in terms of mathematical equations relating ring resonator parameters to the two side waveguides. This leads to the verification of the transfer-matrix relation of the cascade of N resonators. In order to model the cascade of identical resonators in the form of a single complex-valued matrix, the next step is to formally verify the Sylvester's theorem [14]. The final step is to formally define the notion of transmissivity and reflectivity and verify them using the already developed definitions and theorems.

Step 1: In order to describe the cascade of MRRs as a list, we define a new type abbreviation to simplify the reasoning process as follows:

***Definition** 4 (Cascade Microring Resonator (MRR)):*
`new_type_abbrev` **`"mrr_cascade"`** `= :(mrr)list`

We describe the relation among the input and output field parameters (a, b, c, d) in a cascade by splitting them into two pairs: input fields (a_n,d_n) and output fields (b_n,c_n). Here, subscript n represents the field parameters of the n-th resonator. This yields the following definition:

***Definition** 5 (Array of Input and Output Fields):*

```
new_type_abbrev ("single_array",:ℂ × ℂ)
new_type_abbrev ("array",
   :single_array × single_array ×
   (single_array × single_array) list)
```

The first and the second `single_array` represent the pair for input and output fields, respectively. The list of `single_array` pairs represents the same information for the list of MRRs.

Step 2: Although the resonators are uncoupled in the cascade, they are still interacting with the adjacent resonators through the two side waveguides. This interaction is described by the following continuity relations:
$a_{n+1} = b_n * exp(-j * (\frac{2*\pi}{\lambda}) * n_{eff} * L_b)$ and
$c_{n+1} = d_n * exp(j * (\frac{2*\pi}{\lambda}) * n_{eff} * L_b)$

equivalently, as a continuity matrix:

$$\begin{bmatrix} c_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} exp(j(\frac{2\pi}{\lambda})n_{eff} * L_b) & 0 \\ 0 & exp(-j(\frac{2\pi}{\lambda})n_{eff} * L_b) \end{bmatrix} \begin{bmatrix} d_n \\ b_n \end{bmatrix}$$

Next, we define what is the valid behavior of an array within the cascade of uncoupled MRRs periodically coupled with two side waveguides as follows:

***Definition** 6 (Valid Behavior of Array in MRR Cascade):*

```
⊢(valid_array_in_cas (arr:array) []
                          four_port ⇔ F)∧
 (valid_array_in_cas ((a1,d1),(b1,c1),[])
  CONS r (rs:mrr_cascade) four_port ⇔ F)∧
 (valid_array_in_cas
 (a1,d1),(b1,c1), CONS((a2,d2),(b2,c2)) ars
  CONS r rs four_port ⇔
```

$a_2 = b_1 * \exp(-j * (\frac{2*\pi}{\lambda}) * n_{eff} * L_b)\ \wedge$
$c_2 = d_1 * \exp(j * (\frac{2*\pi}{\lambda}) * n_{eff} * L_b)\ \wedge$

```
  is_valid_behavior_in_mrr
  (a1,d1) (b1,c1) r four_port ∧
  valid_array_in_cas ars rs four_port)
```

Here, the predicate `valid_array_in_cas` takes an array (Definition 5), a cascade of MRRs (Definition 4), a type of an MRR structure (which is `four_port` because of the fact that each MRR is coupled to two side waveguides) and returns the corresponding physical behavior. The first two cases are not valid and hence the result is *False*. This is because of the fact that both of the situations: the nonempty array with an empty cascade of MRRs, and the less number of array parameters in case of two or more MRRs do not describe anything physically. Finally, the last case describes the recursive behavior within the cascade using the continuity equations and the predicate `is_valid_behavior_in_mrr` (Definition 3) as described in Section II.

Step 3: We have seen in the previous section that each MRR can be modeled by its corresponding transfer-matrix. In the transfer matrix approach [18], the response of cascade of MRRs coupled to side waveguides can be modeled by the composition of individual matrices of MRRs and the continuity matrix. We model the composition of the cascade of N identical resonators as follows:

***Definition** 7 (Composition of Cascade of MRRs):*
⊢**`cascade_comp`** `[r1;r2;r3;....;rN] =`

$$\prod_{i=1}^{N} \mathtt{continuity\_mat}(r_i)\ ** \ \mathtt{mrr\_mat}(r_i)$$

The functions `continuity_mat` and `mrr_mat` takes a microresonator `r` as an argument and returns the continuity matrix and the matrix of a single MRR derived in Theorem 1, respectively.

Next, we verify one of the most important and generic results in the analysis of MRR based systems [14] which describes the transfer-matrix relation of the cascade of MRRs (Figure 3 (i)).

Theorem *3 (Transfer-Matrix of Cascade of MRRs):*
$\vdash\forall$ arr rrs.
 valid_array_in_cas arr rrs four_port $\Longrightarrow$
 let $((\mathtt{a_1},\mathtt{d_1}),(\mathtt{b_1},\mathtt{c_1}),\mathtt{ars})$ = arr in
 let $\mathtt{c_N},\mathtt{a_N}$ = last_in_out arr

$$\begin{bmatrix} \mathtt{c_N} \\ \mathtt{a_N} \end{bmatrix} = (\texttt{cascade\_comp l\_cas}) ** \begin{bmatrix} \mathtt{c_1} \\ \mathtt{a_1} \end{bmatrix}$$

Here, the parameters arr and rrs represent the array and cascade of MRRs, respectively. last_in_out returns the last input and output field parameter of array in the cascade. The assumption in the above theorem ensures the validity of the good behavior of array in the cascade. Next, we verify the above relation for N identical ring resonators in the cascade as follows:

Theorem *4 (Cascade of Identical MRRs):*
$\vdash\forall$ arr r N.
 valid_array_in_cas arr (REPLICATE N r)
 four_port $\Longrightarrow$
 let $((\mathtt{a_1},\mathtt{d_1}),(\mathtt{b_1},\mathtt{c_1}),\mathtt{ars})$ = arr in
 let $\mathtt{c_N},\mathtt{a_N}$ = last_in_out arr and

$$\text{let } \begin{bmatrix} \mathtt{m_{11}} & \mathtt{m_{12}} \\ \mathtt{m_{21}} & \mathtt{m_{22}} \end{bmatrix} = \texttt{cont\_mat(r)} \; ** \; \texttt{mrr\_mat(r)}$$

$$\begin{bmatrix} \mathtt{c_N} \\ \mathtt{a_N} \end{bmatrix} = \begin{bmatrix} \mathtt{m_{11}} & \mathtt{m_{12}} \\ \mathtt{m_{21}} & \mathtt{m_{22}} \end{bmatrix}^N ** \begin{bmatrix} \mathtt{c_1} \\ \mathtt{a_1} \end{bmatrix}$$

Here, the list $[\mathtt{r};\mathtt{r};\mathtt{r};....;\mathtt{r}]$ of N identical resonators is modeled using the HOL Light library function REPLICATE which takes a natural number N and a parameter (of any type) r and duplicates N copies of r in a list. We verify this theorem using Theorem 3 and the lemma $\prod_{\mathtt{i}=1}^{\mathtt{N}} [\mathtt{M}] = [\mathtt{M}]^{\mathtt{N}}$.

Step 4: Finally, we provide the formal definitions and verification of transmissivity and reflectivity of a cascade of N resonators. The *transmissivity* and *reflectivity* are described as the ratios of output and input field amplitudes of array of MRRs [14] The transmissivity and reflectivity of the cascade of N resonators are $\frac{c_1}{a_1}$ and $\frac{a_N}{a_1}$, respectively. This can be found by the condition $c_N = 0$, which can be defined as follows:

Definition *8 (Transmissivity and Reflectivity Condition):*
$\vdash$ **ref_trans_condition** rrs $\Leftrightarrow$ ($\forall$ arr.
 let $((\mathtt{a_1},\mathtt{d_1}),(\mathtt{b_1},\mathtt{c_1}),\mathtt{ars})$ = arr in
 let $\mathtt{c_N},\mathtt{a_N}$ = last_in_out arr in
 (valid_array_in_cas arr rrs four_port $\wedge$
  $\mathtt{c_N} = 0$)

Here, the predicate ref_trans_condition takes a cascade of microresonators rrs and ensures its valid behavior for any array of the input and the output fields arr and impose the condition that $\mathtt{c_N} = 0$. Next, we verify the general expressions for reflectivity and transmissivity for cascade of MRRs as follows:

Theorem *5 (Transmissivity and Reflectivity Expression):*
$\vdash$ $\forall$ rrs. ref_trans_condition rrs $\Longrightarrow$
 let M = cascade_comp rrs and
 reflect = $\frac{\mathtt{c_1}}{\mathtt{a_1}}$ and **transm** = $\frac{\mathtt{a_N}}{\mathtt{a_1}}$ in
 (reflect = $-\frac{\mathtt{M_{12}}}{\mathtt{M_{11}}}$ $\wedge$ transm = $\frac{1}{\mathtt{M_{11}}}$)

where $\mathtt{M_{ij}}$ represents the element at column i and row j of the matrix. The proof of the above theorem is mainly based on Theorem 3 and the properties of complex matrices and vectors. Note that the result proved in Theorem 4 is very important because it reduces the problem of finding the transmissivity and reflectivity to only finding the equivalent transfer-matrix of the cascade.

IV. Transmissivity and Reflectivity for the 1-D Cascade of MRR Filters

The derivation of transmissivity and reflectivity for the cascade of N identical MRRs is not a trivial task because of the involvement of N-times multiplication of transfer-matrix of a single MRR as given in Theorem 4. However, if the determinant of a resonator matrix $[m]$ is 1 (which is the case in practice [14]), a matrix can be written in such a form that $\exists \mathtt{M}.\ (\mathtt{m})^{\mathtt{N}} = \mathtt{M}$.

This can actually be proved by using Sylvester's Theorem [23], [14], which states that for a matrix m such that $\mid \mathtt{m} \mid = 1$, $-1 < \mathtt{Re}(\mathtt{m_{11}}) < 1$, $\mathtt{m_{22}} = \mathtt{m_{11}^*}$ and $\mathtt{m_{12}} = \mathtt{m_{21}^*}$ the following holds:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^N = \frac{1}{\sin(\theta)} \begin{bmatrix} m_{11}\sin[N\theta] - \sin[(N-1)\theta] & m_{12}\sin[N\theta] \\ m_{21}\sin[N\theta] & m_{22}\sin[N\theta] - \sin[(N-1)\theta] \end{bmatrix}$$

where $\theta = \mathtt{Re}(\mathtt{cos}^{-1}[\frac{\mathtt{m_{11}}+\mathtt{m_{22}}}{2}])$, $\mathtt{Re}(\mathtt{z})$ represents the real part of the complex number z and $\mathtt{a}^*$ represents the conjugate of complex number a. We prove Sylvester's Theorem by induction on N and using the fundamental properties of complex trigonometric functions, complex matrices and determinants. Note that during our formal proof we have been able to find the missing conditions: $-1 < \mathtt{Re}(\mathtt{m_{11}}) < 1$ [14], without which the above result cannot hold. This demonstrates the effectiveness of theorem proving based reasoning about such complex mathematical results. Since the further analysis is based on Sylvester's Theorem, such missing assumptions can lead to erroneous expressions for transmissivity and reflectivity and hence the faulty filter implementation.

Next, we formally verify the reflectivity of a cascade of N identical microring resonators as follows:

Theorem *6 (Transmissivity and Reflectivity for Identical MRR):*
$\vdash\forall$r t $\mathtt{L_c}$ $\mathtt{L_b}$ λ $\mathtt{n_{eff}}$.
  ref_trans_condition
  REPLICATE N $(\mathtt{r},\mathtt{t},\mathtt{L_c},\alpha,\lambda,\mathtt{n_{eff}})$:mrr $\Longrightarrow$
  let $\delta = (\frac{2*\pi}{\lambda}) * \mathtt{n_{eff}} * \mathtt{L_c}$ in
  let $\theta = \mathtt{Re}(\mathtt{cos}^{-1}[\frac{\mathtt{exp}(\mathtt{j}*(\frac{2*\pi}{\lambda})*\mathtt{n_{eff}})}{\mathtt{R}}])$
  reflect = $\frac{\mathtt{T}*\mathtt{exp}(-\mathtt{j}*\frac{\mathtt{L_b}}{\mathtt{L_c}}*\delta)*\mathtt{sin}(\mathtt{N}*\theta)}{\mathtt{exp}(-\mathtt{j}*\frac{\mathtt{L_b}}{\mathtt{L_c}}*\delta)*\mathtt{sin}(\mathtt{N}*\theta)-\mathtt{R}*\mathtt{sin}((\mathtt{N}-1)*\theta)}$
  transm = $\frac{\mathtt{R}}{\mathtt{exp}(-\mathtt{j}*\frac{\mathtt{L_b}}{\mathtt{L_c}}*\delta)*\mathtt{sin}(\mathtt{N}*\theta)-\mathtt{R}*\mathtt{sin}((\mathtt{N}-1)*\theta)}$

The verification of this theorem requires Theorem 4, Theorem 5, Sylvester's theorem and the properties of complex analysis.

This completes the formal analysis of linear cascades of uncoupled resonators periodically coupled to side waveguides.

The analysis of a cascade of coupled resonators (Figure 3(ii)) follows the similar pattern. During our formalization, we have been able to find many discrepancies in the analysis presented in [14], such as the missing assumption in the proof of Sylvester's Theorem. The main strength of theorem proving based analysis is to identify such missing assumptions which needs to be explicitly mentioned in order to build accurate system models. This improved accuracy comes at the cost of the time and efforts spent, while formalizing the underlying theory of microring resonators. But such a developed infrastructure, significantly reduces the time and efforts required to verify important system properties. For example, the verification of transmissivity and reflectivity requires less than 100 lines of HOL code and one man-hour each.

Note that theorem proving based analysis has not been applied in photonic industry so far due to the limited research in this area and unfamiliarity about formal methods in the optics and physics community. In spite of the fact that our approach requires significant time to formalize the underlying theories of optics and photonics, we believe that our formal development can assist in building accurate system models and can replace some time consuming simulations. For example, the verification of transmissivity and reflectivity is a very time consuming task in case of the cascade of microresonators with a very large value of N (for example, $N = 20$ [14]). On the other hand, this can be verified in a very short time using the infrastructure developed in this research because all the results are verified under the universal quantification of system parameters. Mostly, the numerical algorithms are based on such analytical models of photonic systems. Thus the verification of such analytical models can significantly reduce simulation runs due to the known explicit constraints on the systems parameters. Note that the application analyzed in this paper is not a toy example but an advanced photonic system which has been fabricated for different applications [5].

V. Conclusion

In this paper, we report a novel application of formal methods in analyzing microresoantors based photonic systems. We provided a brief introduction of the current state-of-the-art and highlighted their limitations. Next, we presented an overview of microring resonators (MRR) followed by a display of our higher-order logic formalization. We also presented the formalization of frequently used MRR structures such as two-port and four-port. In order to show the practical effectiveness of our formalization, we presented the analysis of 2-D lattice optical filters by decomposing them into two cascades of 1-D coupled and uncoupled resonators. Finally, we developed an infrastructure to verify the transmissivity and reflectivity of the cascade of N identical MRRs.

The reported work opens the doors to many interesting and novel directions of research. Some worth mentioning ones include enriching the library of microresonators to analyze advanced photnic systems such as fiber ring resonators based frequency division multiplexing/demultiplexing [1], ultra compact racetrack resonators [15] and pulse repetition rate shaping for photonic signal processing applications [24]. For example, the application of our work in frequency division multiplexing/demultiplexing [1] requires the formalization (which follows the same steps as mentioned in Section II) of some more building blocks such as directional couplers, single mode fiber channels and fiber-loop reflectors.

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